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VO THI BICH KHUE

OPERATOR CONVEX FUNCTIONS, MATRIX INEQUALITIES AND SOME RELATED TOPICS

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Declaration

This thesis was completed at the Department of Mathematics, Quy Nhon University under the supervision of Assoc. Prof. Dr. Dinh Thanh Duc and Dr. Dinh Trung Hoa. I hereby declare that the results presented in it are new and original. Most of them were published in peer-reviewed journals, others have not been published elsewhere. For using results from joint papers I have gotten permissions from my co-authors.

Binh Dinh, 2018

Vo Thi Bich Khue

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Contents

Glossary of Notation

$\mathbb{C}^n$	:	The linear space of all n-tuples of complex numbers
$\langle x, y \rangle$	:	The scalar product of vectors x and y
$\mathbb{M}_n$	:	The space of $n \times n$ complex matrices
$\mathbb{H}$	:	The Hilbert space
$\mathbb{H}_n$	:	The set of all $n \times n$ Hermitian matrices
$\mathbb{H}_n^+$	:	The set of $n \times n$ positive semi-definite (or positive) matrices
$\mathbb{P}_n$	:	The set of positive definite (or strictly positive) matrices
I, O	:	The identity and zero elements of $\mathbb{M}_n$, respectively
A^*	:	The conjugate transpose (or adjoint) of the matrix A
$\lvert A \rvert$	:	The positive semi-definite matrix $(A^*A)^{1/2}$
$\mathrm{Tr}(A)$	:	The canonical trace of matrix A
$\lambda(A)$	:	The eigenvalue of matrix A
$\sigma(A)$	:	The spectrum of matrix A
$\lVert A \rVert$	:	The operator norm of matrix A
$\lvert\lvert\lvert A \rvert\rvert\rvert$	:	The unitarily invariant norm of matrix A
$x \prec y$	:	x is majorized by y
$A \sharp_t B$	:	The t-geometric mean of two matrices A and B
$A \sharp B$	:	The geometric mean of two matrices A and B
$A \nabla B$	:	The arithmetic mean of two matrices A and B
$A ! B$	:	The harmonic mean of two matrices A and B

$A : B$	:	The parallel sum of two matrices A and B
$M_p(A, B, t)$	:	The matrix p-power mean of matrices A and B
$opgx(p, h, K)$	:	The class of operator (p, h)-convex functions on K
A_+, A_-	:	The positive and the negative parts of matrix A

Introduction

Nowadays, the importance of matrix theory has been well-acknowledged in many areas of engineering, probability and statistics, quantum information, numerical analysis, and biological and social sciences. In particular, positive definite matrices appear as data points in a diverse variety of settings: co-variance matrices in statistics [20], elements of the search space in convex and semi-definite programming [1] and density matrices in the quantum information [72].

In the past decades, matrix analysis becomes an independent discipline in mathematics due to a great number of its applications [5, 7, 18, 24, 25, 26, 27, 34, 39, 46, 85]. Topics of matrix analysis are discussed over algebras of matrices or algebras of linear operators in finite dimensional Hilbert spaces. Algebra of all linear operators in a finite dimensional Hilbert space is isomorphic to the algebra of all complex matrices of the same dimension. One of the main tools in matrix analysis is the spectral theorem in finite dimensional cases. Numerous results in matrix analysis can be transferred to linear operators on infinite dimensional Hilbert spaces without any difficulties. At the same time, many important results from matrices are not true so far for operators in infinite dimensional Hilbert spaces. Recently, many areas of matrix analysis are intensively studied such as theory of matrix monotone and matrix convex functions, theory of matrix means, majorization theory in quantum information theory, etc. Especially, physical and mathematical communities pay more attention on topics of matrix inequalities and matrix functions because of their useful applications in different fields of mathematics and physics as well. Those objects are also important tools in studying operator theory and operator algebra theory as well.

In 1930 von Neumann introduced a mathematical system of axioms of the quantum mechanics as follows:

(i) Each finite dimensional quantum system of n particles is associated with a Hilbert space of dimension 2^n;

(ii) Each observable in such a quantum system corresponds to a Hermitian matrix of the same dimension;

(iii) Each quantum state is associated to a density matrix, i.e., a positive semi-definite matrix of trace 1.

Therefore, matrix theory, matrix analysis and operator theory become the backgrounds of quantum mechanics and hence, several problems in quantum mechanics could be translated to others in the language of matrices. On the other hand, in the last decades along with an intensive development of the quantum information theory, matrix analysis becomes more popular and important.

Recall that if $\lambda_1, \lambda_2, \cdots, \lambda_k$ are eigenvalues of a Hermitian matrix A, then A can be represented as

$$A = \sum_{j=1}^{k} \lambda_j P_j,$$

where P_j is the orthogonal projection onto the subspace spanned by the eigen-vectors corresponding to the eigenvalue λ_j. And for a real-valued function f defined at λ_i ($i = 1, \cdots, k$), the matrix $f(A)$ is well-defined by *the spectral theorem* [43] as

$$f(A) = \sum_{j=1}^{k} f(\lambda_j) P_j. \tag{0.0.1}$$

In quantum theory most of important quantum quantities are defined with the canonical trace Tr on the algebra of matrices. An important quantity is the quantum entropy. For a density matrix A, *the quantum entropy* of A is the value

$$-\mathrm{Tr}(A \log(A)),$$

where the matrix $\log(A)$ is defined by (0.0.1).

It is worth to mention that the function $\log t$ is matrix monotone on $(0, \infty)$, while the function $t \log t$ is matrix convex on $(0, \infty)$. Recall that a function f is operator monotone on $(0, \infty)$ if and only if $tf(t)$ is operator convex on $(0, \infty)$. Operator monotone functions were first studied by K. Loewner in his seminal papers [66] in 1930. In the same decade, F. Krauss introduced operator convex functions [60]. Nowadays, the theory of such functions is intensively studied and becomes an important topic in matrix theory because of their vast of applications in matrix theory and quantum theory as well [41, 54, 55, 57, 63, 65, 69, 73, 75].

In general, a continuous function f defined on $K \subset \mathbb{R}$ is said to be [14]:

- *matrix monotone of order* n if for any Hermitian matrices A and B of order n with spectra in K,
$$A \leq B \quad \Longrightarrow \quad f(A) \leq f(B). \tag{0.0.2}$$

- *matrix convex of order* n if for any Hermitian matrices A and B of order n with spectra in K, and for any $0 \leq \lambda \leq 1$,
$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B). \tag{0.0.3}$$

If the function f is *matrix monotone* (*matrix convex*, respectively) for any dimension of matrices, then it is called *operator monotone* (*operator convex*, respectively).

An important example of operator monotone and convex functions is $f(t) = t^s$. Loewner showed that this function is operator monotone on $\mathbb{R}^+$ if and only if the power $s \in [0, 1]$ while it is operator convex on $(0, \infty)$ if and only if $s \in [-1, 0] \cup [1, 2]$.

Now let us look back at the scalar mean theory which sets a starting point for our study in this thesis.

A *scalar mean* M of non-negative numbers is a function from $\mathbb{R}^+ \times \mathbb{R}^+$ to $\mathbb{R}^+$ such that:

1) $M(x, x) = x$ for every $x \in \mathbb{R}^+$;

2) $M(x, y) = M(y, x)$ for every $x, y \in \mathbb{R}^+$;

3) If $x < y$, then $x < M(x,y) < y$;

4) If $x < x_0$ and $y < y_0$, then $M(x,y) < M(x_0,y_0)$;

5) $M(x,y)$ is continuous;

6) $M(tx,ty) = tM(x,y)$ for $t,x,y \in \mathbb{R}^+$.

A two-variable function $M(x,y)$ satisfying condition 6) can be reduced to a one-variable function $f(x) := M(1,x)$. Namely, $M(x,y)$ is recovered from f as $M(x,y) = xf(x^{-1}y)$. Notice that the function f, corresponding to M is monotone increasing on $\mathbb{R}^+$. And this relation forms a one-to-one correspondence between means and monotone increasing functions on $\mathbb{R}^+$.

In the last few decades, there has been a renewed interest in developing the theory of means for elements in the subset $\mathbb{H}_n^+$ of positive semi-definite matrices in the algebra $\mathbb{M}_n$ of all matrices of order n. Motivated by a study of electrical network connections, Anderson and Duffin [3] introduced a binary operator $A : B$, called parallel addition, for pairs of positive semi-definite matrices. Subsequently, Anderson and Trapp [4] have extended this notion to positive linear operators on a Hilbert space and demonstrated its importance in operator theory. Besides, the problem to find a *matrix analog of the geometric mean of non-negative numbers* was a long-standing problem since the product of two positive semi-definite matrices is not always a positive semi-definite matrix. In 1975, Pusz and Woronowicz [79] solved this problem and showed that the geometric mean $A\sharp B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ of two positive definite matrices A and B is the unique solution of the matrix Riccati equation

$$XA^{-1}X = B.$$

In 1980, Ando and Kubo [61] developed an axiomatic theory of operator means on $\mathbb{H}_n^+$. A binary operation σ on the class of positive operators, $(A,B) \mapsto A\sigma B$, is called a *connection* if the following requirements are fulfilled:

(i) Monotonicity: $A \leq C$ and $B \leq D$ imply $A\sigma B \leq C\sigma D$;

(ii) Transformation: $C^*(A\sigma B)C \leq (C^*AC)\sigma(C^*BC)$;

(iii) Continuity: $A_m \downarrow A$ and $B_m \downarrow B$ imply $A_m\sigma B_m \downarrow A\sigma B$ ($A_m \downarrow A$ means that the sequence A_m converges strongly in norm to A).

A *mean* σ is a connection satisfying the normalized condition:

(iv) $I\sigma I = I$ (where I is the identity element of $\mathbb{M}_n$).

The main result in Kubo-Ando theory is the proof of the existence of an affine order-isomorphism from the class of operator means onto the class of positive operator monotone functions on $\mathbb{R}^+$ which is described by

$$A\sigma_f B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}.$$

This formula verifies that the geometric mean defined by Pusz and Woronowicz was natural and corresponding to the operator monotone function $f(t) = t^{1/2}$. A mean σ is called *symmetric* if $A\sigma B = B\sigma A$ for any positive matrices A and B. Or, equivalently, the representing function f of a symmetric mean satisfies $f(t) = tf(t^{-1})$, $t \in (0, \infty)$.

Later, motivated by information geometry, Morozova and Chentsov [69] studied monotone inner products under stochastic mappings on the space of matrices and monotone metrics in quantum theory. In 1996, Petz [78] proved that there is a correspondence between monotone metrics and operator means in the sense of Kubo and Ando, and hence, connected three important theories in quantum information theory and matrix analysis.

It is worth to mention that along with the quantum entropy of quantum states, many other important quantum quantities are defined with operator means, operator convex functions and the canonical trace.

Example 0.0.1. For two density matrices A and B, the *quantum relative entropy* [64] of A with respect to B is defined by

$$S(A||B) = -\operatorname{Tr}(A(\log A - \log B)).$$

The *quantum Chernoff bound* [10] in quantum hypothesis testing theory is given by a simple expression: For positive semi-definite matrices A and B,

$$Q(A, B) = \min_{0 \le s \le 1} \{\mathrm{Tr}(A^s B^{1-s})\}.$$

One of important quantities in quantum theory is the *Renyi divergence* [20]: for $\alpha \in (0, 1) \cup (1, \infty)$,

$$D_\alpha(A||B) = \frac{1}{\alpha - 1} \log \frac{\mathrm{Tr}(A^s B^{1-s})}{\mathrm{Tr}(A)}, \quad D_1 = \frac{\mathrm{Tr}(A(\log A - \log B))}{\mathrm{Tr}(A)}.$$

All of quantities listed above are special cases of the quantum f-divergence in quantum theory where f is some operator convex function [45]. Thus, the theory of matrix functions is an important part of matrix analysis and of quantum information theory as well.

Now let σ and τ be arbitrary operator means (not necessarily Kubo-Ando means) [61]. We introduce a general approach to operator convexity as follows.

A non-negative continuous function f on $\mathbb{R}^+$ is called *$\sigma\tau$-convex* if for any positive definite matrices A and B,

$$f(A \sigma B) \le f(A) \tau f(B). \tag{0.0.4}$$

When σ and τ are the arithmetic mean, the function f satisfying the above inequality is operator convex. When σ is the arithmetic mean and τ is the geometric mean, the function f satisfying (0.0.4) is called *operator* $\log$*-convex*. Such functions were fully characterized by Hiai and Ando in [11] as decreasingly monotone operator functions.

The matrix power mean of positive semi-definite matrices A and B was first studied by Bhagwat and Subramanian [15] as

$$M_p(A, B, t) = (tA^p + (1-t)B^p)^{1/p}, \quad \text{for} \quad p \in \mathbb{R}.$$

The matrix power mean $M_p(A, B, t)$ is a Kubo-Ando mean if and only if $p = \pm 1$. Nevertheless, the power means with $p > 1$ have many important applications in mathematical physics and in the theory of operator spaces [21].

In this thesis, we use (0.0.4) to define some new classes of operator convex functions with the matrix power means $M_p(A, B, t)$. We study properties of such functions and prove some well-known inequalities for them. We also provide several equivalent conditions for a function to be operator convex in this new sense.

Now, let us consider some geometrical interpretations for scalar means and matrix means. Let $0 \leq a \leq x \leq b$. It is obvious that the arithmetic mean $(a+b)/2$ is the unique solution of the optimization problem

$$(x-a)^2 + (x-b)^2 \to \min, \quad x \in \mathbb{R}.$$

And for any scalar mean M on $\mathbb{R}^+$,

$$M(a, b) - a \leq b - a.$$

We call this *the in-betweenness property*.

In 2013, Audenaert studied the in-betweenness property for matrix means in [9]. Recently, Dinh, Dumitru and Franco [49] continued to investigate this property for the matrix power means. They provided some partial solutions to Audenaert's conjecture in [9] and a counterexample to the conjecture for $p > 0$.

From the property 3) in the definition of scalar means, it is obvious that,

$$\frac{a+b}{2} - M(a, b) \leq \frac{b-a}{2}. \tag{0.0.5}$$

In other words, $M(a, b)$ lies inside the sphere centered at the arithmetic mean $\dfrac{a+b}{2}$ with the radius equal to a half of the distance between a and b. We call this *the in-sphere property* of scalar means with respect to the Euclidean distance on $\mathbb{R}$. Notice that for $s \in [0, 1]$ and $p > 0$ the s-weighted geometric mean $M(a, b) = a^{1-s}b^s$ and the power mean (or binomial mean) $M_p(a, b, s) = ((1-s)a^p + sb^p)^{1/p}$ satisfy the in-sphere property (0.0.5).

Now, let A and B be positive definite matrices. The Riemannian distance function on the set of positive definite matrices is defined by

$$\delta_R(A,B) = \left(\sum_i \log^2(\lambda_i(A^{-1}B)) \right)^{1/2}.$$

In 2005, Moakher [67] showed that the geometric mean $A\sharp B$ is the unique minimizer of the sum of the squares of the distances:

$$\delta_R^2(X,A) + \delta_R^2(X,B) \to \min, \quad X \geq 0.$$

Almost at the same time, Bhatia and Holbrook [17] showed that the curve

$$\gamma(s) = A\sharp_s B := A^{1/2}(A^{-1/2}BA^{-1/2})^s A^{1/2} \quad (s \in [0,1])$$

is the unique geodesic (i.e., the shortest) path joining A and B. Furthermore, the geometric mean $A\sharp B$ is the midpoint of this geodesic. Therefore, the picture for matrix means is very different from the one for scalar ones.

Notice that one of the important matrix generalizations of the in-sphere property is the famous Powers-Størmer inequality proved by Audenaert et. al. [10], and then expanded to operator algebras by Ogata [74]: for any positive semi-definite matrices and for any $s \in [0,1]$,

$$\operatorname{Tr}(A + B - |A - B|) \leq 2\operatorname{Tr}(A^s B^{1-s}). \tag{0.0.6}$$

Using the last inequality the authors solved a problem in quantum hypothesis testing theory: to define the quantum generalization of the Chernoff bound [23]. The quantity on the left hand side of (0.0.6) is called *the non-logarithmic quantum Chernoff bound*. Along with the mentioned above importance of matrix means, the Powers-Størmer inequality again shows us that the picture of matrix means is really interesting and complicated.

The second aim of this thesis is to investigate various matrix versions of in-sphere property

(0.0.5). More precisely, we study inequalities involving matrices, matrix means, trace, norms and matrix functions. We also consider the in-sphere property for matrix means with respect to some distance functions on the manifold of positive semi-definite matrices.

The purposes of the current thesis are as follows.

1. Investigate new types of operator convex functions with respect to matrix means, study their properties and prove some well-known inequalities for them.

2. Characterize new types of operator convex functions by matrix inequalities.

3. Study reverse arithmetic-geometric means inequalities involving general matrix means.

4. Study reverse inequalities for the matrix Heinz means and unitarily invariant norms.

5. Study in-sphere properties for matrix means with respect to unitarily invariant norms.

Methodology. The main tool in our research is the spectral theorem for Hermitian matrices. We use techniques in the theory of matrix means of Kubo and Ando to define new types of operator convexity. Some basic techniques in the theory of operator monotone functions and operator convex functions are also used in the dissertation. We also use basic knowledge in matrix theory involving unitarily invariant norms, trace, etc.

Main results of the work were presented on the seminars at the Department of Mathematics at Quy Nhon University and on international workshops and conferences as follows:

1. The Second Mathematical Conference of Central and Highland of Vietnam, Da Lat University, November 2017.
2. The 6th International Conference on Matrix Analysis and Applications (ICMAA 2017), Duy Tan University, June 2017.
3. Conference on Algebra, Geometry and Topology (DAHITO), Dak Lak Pedagogical College, November 2016.
4. International Workshop on Quantum Information Theory and Related Topics, VIASM, September 2015.
5. Conference on Mathematics of Central-Highland Area of Vietnam, Quy Nhon University, August 2015.

6. Conference on Algebra, Geometry and Topology (DAHITO), Ha Long, December 2014.
7. International Workshop on Quantum Information Theory and Related Topics, Ritsumeikan University, Japan, September 2014.

This thesis has Introduction, three chapters, Conclusion, a list of the author's papers related to the thesis and preprints related to the topics of the thesis, and a list of references.

Brief content of the thesis.

In Introduction the author provides a background on the topics which are considered in this work. The meaningfulness and motivations of this work are explained. The author also provides a brief content of the thesis with main results from the last two chapters.

In the first chapter the author collects some basic preliminaries which are used in the thesis.

In the second chapter the author defines and studies new classes of operator convex functions, their properties, proves some well-known inequalities for them and obtains a series of characterizations.

Let $\mathbb{M}_n$ be the space of $n \times n$ complex matrices, $\mathbb{H}_n$ the set of all $n \times n$ Hermitian matrices and $\mathbb{H}_n^+$ the set of positive semi-definite matrices in $\mathbb{M}_n$. In this work, *we always assume that p is some positive number, J is an interval in $\mathbb{R}^+$ such that* $(0,1) \subset J$. The set K ($\subset \mathbb{R}^+$) is always a p-convex set (i.e., $[\alpha x^p + (1-\alpha)y^p]^{1/p} \in K$ for all $x, y \in K$ and $\alpha \in [0,1]$), and h is an *super-multiplicative function* on J (i.e., $h(xy) \geq h(x)h(y)$ for any x and y in J).

Definition 2.1.2 ([51]). Let $h : J \to \mathbb{R}^+$ be a super-multiplicative function. A non-negative function $f : K \to \mathbb{R}$ is said to be *operator (p,h)-convex* (or belongs to the class $opgx(p,h,K)$) if for any $n \in \mathbb{N}$ and for any $A, B \in \mathbb{H}_n^+$ with spectra in K, and for $\alpha \in (0,1)$, we have

$$f\left([\alpha A^p + (1-\alpha)B^p]^{1/p}\right) \leq h(\alpha)f(A) + h(1-\alpha)f(B). \tag{2.1.4}$$

When $p = 1,\ h(\alpha) = \alpha$, we get the usual definition of operator convex functions on $\mathbb{R}^+$.

The class of operator (p,h)-convex functions contains several well-known classes of functions such as non-negative convex functions, h- and p-convex functions [13], Godunova-Levin functions (or Q-class functions) [30] and P-class functions [70]. An operator (p,h)-convex function could be either an operator monotone function or an operator convex function. On the other hand, many power functions are operator (p, h)-convex but are neither an operator monotone nor an operator convex.

Operator (p, h)-convex functions satisfy some properties. Besides, we also obtain matrix versions of Jensen type inequality, Hansen-Pedersen type inequality for operator (p, h)-convex functions. And finally, we provide a series of equivalent conditions for a continuous function to be operator (p, h)-convex.

Theorem 2.1.6 ([51]). *Let f be a non-negative function on the interval K such that $f(0) = 0$, and h a non-negative and non-zero super-multiplicative function on J satisfying $2h(1/2) \leq \alpha^{-1}h(\alpha)$ $(\alpha \in (0, 1))$. Then the following statements are equivalent:*

(i) *f is an operator (p, h)-convex function;*

(ii) *for any contraction V ($\|V\| \leq 1$) and self-adjoint matrix A with spectrum in K,*

$$f((V^*A^pV)^{1/p}) \leq 2h(1/2)V^*f(A)V;$$

(iii) *for any orthogonal projection Q and any Hermitian matrix A with spectrum in K,*

$$f((QA^pQ)^{1/p}) \leq 2h(1/2)Qf(A)Q;$$

(iv) *for any natural number k, for any families of positive operators $\{A_i\}_{i=1}^k$ in a finite dimensional Hilbert space $\mathbb{H}$ satisfying $\sum_{i=1}^k \alpha_i A_i = I_{\mathbb{H}}$ (the identity operator in $\mathbb{H}$) and for arbitrary numbers $x_i \in K$,*

$$f\left(\left[\sum_{i=1}^k \alpha_i x_i^p A_i\right]^{1/p}\right) \leq \sum_{i=1}^k h(\alpha_i)f(x_i)A_i. \tag{2.1.15}$$

In the second section of this chapter we define another type of convexity which is called operator (r, s)-convexity.

For a pair $X = (A_1, A_2)$ of Hermitian matrices with $\sigma(A_1), \sigma(A_2) \subset K$, and a function f, we define $f(X) = (f(A_1), f(A_2))$. For a pair of positive numbers $W = (\omega_1, \omega_2)$, we set $W_2 := \omega_1 + \omega_2$ and define the weighted matrix r-power mean $M^{[r]}(X, W)$ to be

$$M^{[r]}(X, W) := \left[\frac{1}{W}(\omega_1 A_1^r + \omega_2 A_2^r)\right]^{1/r}.$$

Definition 2.2.1 ([48]). Let r, s be arbitrary numbers, and K be an interval in $\mathbb{R}^+$. A continuous function $f : K \to (0, \infty)$ is said to be *operator* (r, s)*-convex* if

$$f(M^{[r]}(X, W)) \leq M^{[s]}(f(X), W). \tag{2.2.16}$$

where $X, W, f(X), M^{[r]}(X, W)$ are defined as above.

We obtain some properties of operator (r, s)-convex functions which are similar to those of operator (p, h)-convex functions. We also prove the Rado inequality for such functions.

In the third chapter, we study the in-sphere property for matrix means. We also establish some reverse inequalities for the matrix Heinz means and provide a new characterization of the matrix arithmetic mean.

Firstly, notice that for two non-negative numbers a and b and for any number $s \in [0, 1]$, it is obvious that

$$\min\{a, b\} = \frac{a+b}{2} - \frac{|a-b|}{2} \leq a^{1-s}b^s = a\sharp_s b. \tag{3.0.2}$$

and the following inequality for the Heinz mean $H_s(a, b) = \dfrac{a^s b^{1-s} + a^{1-s}b^s}{2}$ is an immediate consequence of (3.0.2)

$$\frac{a+b}{2} - \frac{|a-b|}{2} \leq \frac{a^{1-s}b^s + a^s b^{1-s}}{2}. \tag{3.0.3}$$

And the arithmetic-geometric means (AGM) inequality has a refinement given by

$$\sqrt{ab} \leq \frac{a^{1-s}b^s + a^s b^{1-s}}{2} \leq \frac{a+b}{2} \tag{3.0.4}$$

Various matrix generalizations of inequality (3.0.2) and (3.0.3) are a timely research subject under active investigation.

In the first section of the third chapter, for some symmetric operator mean σ (i.e., $A\sigma B = B\sigma A$ for any positive definite matrices A, B) we investigate a matrix version of (3.0.2) and (3.0.3) of the form:

$$\frac{A+B}{2} - \frac{1}{2}|A-B| \leq A\sigma_f B. \tag{3.1.6}$$

We show that generalized reverse AGM inequalities hold under the extra condition that $AB+BA$ is positive semi-definite.

Theorem 3.1.1 ([50]). *Let f be a strictly positive operator monotone function on $[0, \infty)$ with $f((0,\infty)) \subset (0,\infty)$ and $f(1) = 1$. Then for any positive semi-definite matrices A and B with $AB + BA \geq 0$,*

$$\frac{A+B}{2} - \frac{1}{2}|A-B| \leq A\sigma_f B. \tag{3.1.7}$$

By using Matlab, we provide counter-examples which confirm that the condition $AB + BA \geq 0$ is essential.

In the second section we study some reverse inequalities to (3.0.4) for unitarily invariant norms and the matrix Heinz mean. Recall that a norm $|||\cdot|||$ on $\mathbb{M}_n$ is unitarily invariant if $|||UAV||| = |||A|||$ for any unitary matrices U, V and any $A \in \mathbb{M}_n$.

Theorem 3.2.1 ([52]). *Let $|||\cdot|||$ be an arbitrary unitarily invariant norm on $\mathbb{M}_n$. Let f be an operator monotone function on $[0,\infty)$ with $f((0,\infty)) \subset (0,\infty)$ and $f(0) = 0$, and g a function* *on* $[0,\infty)$ *such that* $g(t) = \frac{t}{f(t)}$ $(t \in (0,\infty))$ *and* $g(0) = 0$. *Then for any* $A, B \in \mathbb{P}_n$,

$$\left|\left|\left|\frac{A+B}{2} - \frac{1}{2}A^{1/2}|I_n - A^{-1/2}BA^{-1/2}|A^{1/2}\right|\right|\right| \le \left|\left|\left|f(A)^{1/2}g(B)f(A)^{1/2}\right|\right|\right|$$
$$\le |||f(A)g(B)|||.$$

As an application, for $f(t) = t^s$ $(s \in [0,1])$, we obtain a reverse inequality for the matrix Heinz mean as in Theorem 3.2.2 and some inequalities for unitarily invariant norms.

In the rest of the thesis we consider the in-sphere property for operator means. Firstly, recall that from (3.1.7), for any operator mean σ, for any unitarily invariant norm $|||\cdot|||$ and for any $A, B \in \mathbb{H}_n^+$ with $AB + BA \ge 0$, we have

$$\left|\left|\left|\frac{A+B}{2} - A\sigma B\right|\right|\right| \le \frac{1}{2}|||A-B|||.$$

The last inequality means that whatever operator mean σ we take, the point $A\sigma B$ can not run out of the sphere with center at $\dfrac{A+B}{2}$ and the radius equal to $\dfrac{1}{2}|||A-B|||$. This is one of matrix versions of in-sphere property of operator means. However, if we fix some operator mean σ which is different from the arithmetic mean, then we can find a couple of matrices A, B so that $A\sigma B$ runs away from the sphere mentioned aboved. In the next theorem, we provide a new characterization of the matrix arithmetic mean by the inequality (3.1.7).

Theorem 3.3.1 ([52]). *Let* σ *be an arbitrary symmetric mean. If for an arbitrary unitarily invariant norm* $|||\cdot|||$ *on* $\mathbb{M}_n$,

$$\left|\left|\left|\frac{A+B}{2} - A\sigma B\right|\right|\right| \le \frac{1}{2}|||A-B||| \tag{3.3.24}$$

whenever $A, B \in \mathbb{P}_n$, *then* σ *is the arithmetic mean.*

Finally, we show that if we replace the Kubo-Ando means by the power mean $M_p(A,B,t)$ $= (tA^p + (1-t)B^p)^{1/p}$ with $p \in [1,2]$ then the inequality in Theorem 3.3.1 holds without the condition $AB + BA \ge 0$. In other words, the matrix power means $M_p(A,B,t)$ satisfies the in-sphere property with respect to the Hilbert-Schmidt 2-norm.

Theorem 3.3.2 ([52]). *Let $p \in [1,2]$ and $M_p(A,B,t) = (tA^p + (1-t)B^p)^{1/p}$. Then for any pair of positive semi-definite matrices A and B, we have*

$$\left\| \frac{A+B}{2} - M_p(A,B,t) \right\|_2 \leq \frac{1}{2} \left\| A-B \right\|_2. \tag{3.3.28}$$

Chapter 1

Preliminaries

Let $\mathbb{N}$ be the set of all natural numbers. For each $n \in \mathbb{N}$, we denote by $\mathbb{M}_n$ the algebra of all $n \times n$ complex matrices. Denote by I and O the identity and zero elements of $\mathbb{M}_n$, respectively.

In this thesis we consider problems for matrices, i.e., operators in finite dimensional Hilbert spaces. We will mention if the case is infinite dimensional.

Recall that for two vectors $x = (x_j), y = (y_j) \in \mathbb{C}^n$ the *inner product* $\langle x, y\rangle$ of x and y is defined as $\langle x, y\rangle \equiv \sum_j x_j \overline{y}_j$. Now let A be a matrix in $\mathbb{M}_n$. The *conjugate transpose* or the *adjoint* A^* of A is the complex conjugate of the transpose A^T. We have, $\langle Ax, y\rangle = \langle x, A^* y\rangle$.

A matrix A is called:

– *self-adjoint* or *Hermitian* if $A = A^*$, or, it is equivalent to that $\langle Ax, y\rangle = \langle x, Ay\rangle$;

– *unitary* if $AA^* = A^*A = I$;

– *normal* if $AA^* = A^*A$;

– *positive semi-definite* (or positive) (we write $A \geq 0$) if

$$\langle x, Ax\rangle \geq 0 \quad \text{for all} \quad x \in \mathbb{C}^n; \tag{1.0.1}$$

– *positive definite* (or strictly positive) (we write $A > 0$) if (1.0.1) is strict for all non-zero vector $x \in \mathbb{C}^n$;

– *orthogonal projection* if $A = A^* = A^2$.

Note that in the finite dimensional case, $A > 0$ if and only if A is invertible and $A \geq 0$. A positive semi-definite matrix is necessary Hermitian. Further, we denote by $\mathbb{H}_n$ the set of all $n \times n$ Hermitian matrices, by $\mathbb{H}_n^+$ and $\mathbb{P}_n$ the $n \times n$ positive semi-definite and positive definite matrices, respectively.

Lemma 1.0.1. *The following statements are equivalent:*

(i) *A is positive semi-definite;*

(ii) *A is Hermitian and all its eigenvalues are non-negative;*

(ii) *$A = B^*B$ for some matrix B;*

(iii) *$A = T^*T$ for some upper triangular T;*

(iv) *$A = T^*T$ for some upper triangular T. Moreover, T can be chosen to have non-negative diagonal entries. If A is positive definite, then T is unique*[1]*;*

(v) *$A = B^2$ for some positive semi-definite matrix B. Such a B is unique, denoted by $B = A^{1/2}$ and called the (positive) square root of A.*

The matrix A is positive definite if and only if B is positive definite.

Notice that for any matrix A, the matrix A^*A is always positive semi-definite. Hence, as a consequence of *(v)*, *the module* $|A|$ of A is well defined to be $|A| := (A^*A)^{1/2}$.

Now let us define a partial order on the set $\mathbb{H}_n$ of Hermitian matrices as follows:

$$A \geq B \quad \text{if} \quad A - B \geq 0.$$

This is known as the *Loewner partial order*.

The *canonical trace* of a matrix $A = (a_{ij}) \in \mathbb{M}_n$, denoted by $\mathrm{Tr}(A)$, is the sum of all diagonal entries, or, we often use the sum of all eigenvalues $\lambda_i(A)$ of A, i.e.,

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i(A).$$

[1] This is called the *Cholesky Decomposition of A*

A positive semi-definite matrix A with trace 1 is called *a density matrix* which is associated with a quantum state in some quantum system. In this sense, all rank one orthogonal projections in $\mathbb{M}_n$ are called *pure states*. And positive semi-definite matrices are called *mixed states*.

For a matrix/operator A, the *operator norm* of A is defined as

$$\|A\| = \sup\{\|Ax\| : x \in \mathbb{H}, \|x\| \leq 1\}$$

where $\|x\| = \langle x, x\rangle^{1/2}$.

An operator A is called a *contraction* if $\|A\| \leq 1$.

One of the most important information about operators/matrices are their *spectra*. Generally, the spectrum $\sigma(A)$ of a linear operator A acting in some Hilbert space consists of all numbers $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is not invertible. Therefore, in the finite dimensional case *the spectrum* $\sigma(A)$ of a matrix A is the set of eigenvalues of A, i.e., all numbers λ such that $Ax = \lambda x$. Eigenvalues $s_i(A)$ of the module $|A|$ are called *the singular values* (also called s-numbers) of A. For a matrix $A \in \mathbb{M}_n$, the notation $s(A) \equiv (s_1(A), s_2(A), ..., s_n(A))$ means that $s_1(A) \geq s_2(A) \geq ... \geq s_n(A)$.

Now let us recall some important norms which will be considered in this thesis.

The *Ky Fan k-norm* is the sum of all singular values, i.e.,

$$||A||_k = \sum_{i=1}^{k} s_i(A).$$

The *Schatten p-norm* is defined as

$$||A||_p = \left(\sum_{i=1}^{n} s_i^p(A)\right)^{1/p}.$$

When $p = 2$, we have the *Frobenius norm* or sometimes called the *Hilbert-Schmidt norm* :

$$\|A\|_2 = (\mathrm{Tr}\,|A|^2)^{1/2} = \left(\sum_{j=1}^{n} s_j^2(A)\right)^{1/2}.$$

Let $x = (x_1, ..., x_n)$ be an element of $\mathbb{R}^n$. Let $x = (x_{[1]}, ..., x_{[n]})$ be the vector obtained by rearranging the coordinates of x in the decreasing order ($x_{[1]} \geq x_{[2]} \geq ... \geq x_{[n]}$).

Let $x, y \in \mathbb{R}^n$. If

$$\Sigma_{i=1}^{k} x_{[i]} \leq \Sigma_{i=1}^{k} y_{[i]}, k = 1, 2, ..., n,$$

then we say x is *weakly majorized* by y and denote $x \prec_w y$.

If in addition to $x \prec_w y, \Sigma_{i=1}^{k} x_{[i]} = \Sigma_{i=1}^{k} y_{[i]}$ holds, then we say that x is *majorized* by y and denote $x \prec y$.

Example 1.0.1. If each $a_i \geq 0, \Sigma_{i=1}^{n} a_i = 1$ then

$$\left(\frac{1}{n}, ..., \frac{1}{n}\right) \prec (a_1, ..., a_n) \prec (1, 0, ..., 0).$$

We call a matrix *non-negative* if all its entries are non-negative real numbers. A non-negative matrix is called *doubly stochastic* if all its row and column sums are one.

Definition 1.0.1. A norm $|||\cdot|||$ on $\mathbb{M}_n$ is called *unitarily invariant* if

$$|||UAV||| = |||A|||$$

for any matrix $A \in \mathbb{M}_n$ and for any unitary matrices $U, V \in \mathbb{M}_n$.

It is well-known that every unitarily invariant norm is *sub-multiplicative* [16, p. 94]:

$$|||AB||| \leq |||A||| \cdot |||B||| \qquad \text{for all } A, B.$$

If the product AB is normal, then for every unitarily invariant norm, we have [16, p. 253]

$$|||AB||| \leq |||BA|||.$$

Now let us recall the spectral theorem which is one of the most important tools in functional analysis and matrix theory. In mathematics, particularly in linear algebra and functional analysis, the spectral theorem is a result about the diagonalizability of linear operators.

Theorem 1.0.1 (Spectral Decomposition)**.** *Let $\lambda_1 > \lambda_2 ... > \lambda_k$ be eigenvalues of a Hermitian matrix A. Then*

$$A = \sum_{j=1}^{k} \lambda_j P_j, \tag{1.0.2}$$

where P_j is the orthogonal projection onto the subspace spanned by the eigenvectors associated to the eigenvalue λ_j.

The formula (1.0.2) is called *the spectral decomposition* of A.

For a real-valued function f defined on some interval K and for a self-adjoint matrix $A \in \mathbb{M}_n$ with spectrum in K, the matrix $f(A)$ is defined by means of the functional calculus, i.e.,

$$A = \sum_{j=1}^{k} \lambda_j P_j \qquad \Longrightarrow \qquad f(A) := \sum_{j=1}^{k} f(\lambda_j) P_j. \tag{1.0.3}$$

In another words, if $A = U\text{diag}(\lambda_1, ..., \lambda_n)U^*$ is a spectral decomposition of A (where U is some unitary), then

$$f(A) := U\text{diag}(f(\lambda_1), \cdots, f(\lambda_n))U^*. \tag{1.0.4}$$

Let A be a Hermitian matrix with the spectral decomposition $A = \sum_{i=1}^{k} \lambda_i P_i$. Then its positive part A_+ and negative part A_- are defined as follows:

$$A_+ = \sum \lambda_i P_i \quad \text{with} \quad \lambda_i > 0,$$

and

$$A_- = -\sum \lambda_j P_j \quad \text{with} \quad \lambda_j < 0.$$

It implies

$$A = A_+ - A_-$$

. Now we are at the stage to discuss about matrix/operator functions. Operator monotone functions were first studied by Loewner in his seminal papers [66] in 1930. In the same decade, Krauss introduced operator convex functions [60]. Nowadays, the theory of such functions is intensively studied and becomes an important topic in matrix theory because of their vast of applications in matrix theory and quantum theory as well.

Definition 1.0.2. A continuous function f defined on an interval K ($K \subset \mathbb{R}$) is said to be *matrix monotone of order* n on K if for any Hermitian matrices A and B in $\mathbb{M}_n$ with spectra in K,

$$A \leq B \quad \text{implies} \quad f(A) \leq f(B).$$

If f is matrix monotone of any orders then f is called *operator monotone.*

Definition 1.0.3. A continuous function f defined on an interval K ($K \subset \mathbb{R}$) is said to be *matrix convex of order* n on K if for any Hermitian matrices A and B in $\mathbb{M}_n$ with spectra in K and for all real numbers $0 \leq \lambda \leq 1$,

$$f\big(\lambda A + (1-\lambda)B\big) \leq \lambda f(A) + (1-\lambda) f(B). \tag{1.0.5}$$

If f is matrix convex of any orders then f is called *operator convex*. It is worth to mention that if the eigenvalues of A and B are all in an interval K, then the eigenvalues of any convex combination of A, B are also in K.

One of the very important examples of operator monotone functions and operator convex functions is the function $f(t) = t^r$. Loewner [66] showed that this function is operator monotone on $\mathbb{R}^+$ if and only if the power $r \in [0, 1]$, i.e.,

$$0 \leq A \leq B \quad \Longrightarrow \quad A^r \leq B^r.$$

The last inequality is well-known as *Loewner-Heinz inequality*. When $r \in [-1, 0] \cup [1, 2]$ the power function is operator convex, i.e., for any positive semi-definite matrices A, B and for any $\lambda \in [0, 1]$,

$$(\lambda A + (1-\lambda)B)^r \leq \lambda A^r + (1-\lambda)B^r.$$

Another important example is the function $f(t) = \log t$, which is operator monotone on $(0, \infty)$ and the function $g(t) = t \log t$ is operator convex. These two functions appear in the definition of quantum entropy of states in quantum information theory. For a density matrix A, the quantum entropy of A is defined as

$$-\operatorname{Tr}(A \log A).$$

In the formulation of the spectral theorem the function f only needs to be defined at eigenvalues of the matrix A and not necessary to be continuous. But *in this thesis, we will consider only continuous functions*. In this case, the operator convexity (1.0.5) of the function f is equivalent to

$$f\left(\frac{A+B}{2}\right) \leq \frac{f(A)+f(B)}{2}. \tag{1.0.6}$$

Functions satisfying (1.0.6) are called *mid-point operator convex*.

It is clear that the set of operator monotone functions and the set of operator convex functions are both closed under positive linear combinations and also under pointwise limits. In other words, if f, g are operator monotone, and if α, β are positive real numbers, then $\alpha f + \beta g$ is also operator monotone. If $\{f_m\}$ are operator monotone, and if $f_m(x) \to f(x)$, then f is also operator monotone. The same is true for operator convex functions.

Matrix monotone/convex functions are monotone/convex in the usual sense. The opposite is not true. For example, the function t^2 is not operator monotone on $\mathbb{R}^+$. Indeed, let $A, B \in \mathbb{H}_n^+$,

$$(A+B)^2 = A^2 + (AB + BA) + B^2.$$

For any choice of A and B such that $AB + BA$ has even one strictly negative eigenvalue, the inequality $(A + tB)^2 \geq A^2$ will fail for all sufficiently small t. It is easy to find such A and B in $\mathbb{H}_n^+$. For example, take

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

so that

$$AB + BA = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

The monotonicity and the concavity of functions in real analysis are different. However, they are the same for operator case considered on the positive half-line $[0, \infty)$. A continuous function f is operator monotone on $[0, \infty)$ if and only if it is operator concave.

It is well-known that a continuous function f on the interval $(0, \alpha)$ is operator convex if and only if the function $g(t) = f(t)/t$ is operator monotone on $(0, \alpha)$. In finite dimensional case we have the following tight relation between matrix monotone functions and matrix convex functions on $\mathbb{R}^+$.

Theorem 1.0.2 ([53])**.** *Let f be a strictly positive, continuous function on $[0, \infty)$. If f is monotone of order $2n$, the function $g(t) = \frac{t}{f(t)}$ is monotone of order n on $[0, \infty)$.*

As a consequence, it follows from the above theorem that a such relation is still valid for operator monotone functions and operator convex functions on $\mathbb{R}^+$.

The theory of operator monotone functions has been paid more attention after a series of important papers on operator means and their applications. Motivated by a study of electrical network connections, Anderson and Duffin [3] introduced a binary operator $A : B$, called *the parallel addition* for pairs of positive matrices as

$$A : B = (A^{-1} + B^{-1})^{-1}.$$

The harmonic mean is $2(A : B)$ which is the dual of the arithmetic mean. In the same year, Pusz and Woronowicz [79] considered a binary operation

$$A \sharp B := A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2},$$

called *the geometric mean* of two positive definite matrices A and B. They showed that the geometric mean is the unique solution of the Riccati matrix equation

$$XA^{-1}X = B,$$

and hence, solved a problem how to define the geometric mean of two positive semi-definite matrices.

In 1980, Ando and Kubo [61] developed an axiomatic theory of *operator mean* . A binary operation σ on the class of positive operators, $(A, B) \mapsto A\sigma B$, is called a *connection* if the following requirements are fulfilled:

(i) $A \leq C$ and $B \leq D$ imply $A\sigma B \leq C\sigma D$.

(ii) $C^*(A\sigma B)C \leq (C^*AC)\sigma(C^*BC)$.

(iii) $A_m \downarrow A$ and $B_m \downarrow B$ imply $A_m\sigma B_m \downarrow A\sigma B$ (where $A_m \downarrow A$ means that the sequence A_m converges strongly in norm to A).

A *mean* is a connection σ satisfying the normalized condition:

(iv) $I\sigma I = I$.

An immediate consequence of (ii) is

(ii)$_0$ $C(A\sigma B)C = (CAC)\sigma(CBC)$ for any invertible operator C.

In particular, every connection is positively homogeneous, i.e.,

$$a(A\sigma B) = (aA)\sigma(aB) \text{ for } a > 0.$$

The key result in the theory of Kubo and Ando is that operator means are in a 1-to-1 correspondence with operator monotone functions on $[0, \infty)$ and that correspondence is given by

$$A\sigma_f B = A^{1/2} f(A^{-1/2}BA^{-1/2})A^{1/2}, \quad A, B \in \mathbb{H}_n^+. \tag{1.0.7}$$

The function f is called *representing function* of σ.

Note that in (1.0.7) the matrix A should be invertible. But we can still define operator means for positive semi-definite matrices as follows: For $A, B \geq 0$ and for an operator mean σ, we can define $(A + \varepsilon I)\sigma(B + \varepsilon I)$ with $\varepsilon > 0$. Then, by the continuity of the representing function of σ, letting ε tend to zero from the right we get $A\sigma B$. Further, without any special mention we always consider operator means for positive semi-definite matrices.

Notice that the Kubo-Ando theory was built based on the integral representation of operator monotone functions. For matrix monotone functions there is no such representations. Therefore, if we restrict our consideration on means with respect to the matrix algebra of a fixed order, some questions become extremely difficult. Since in this thesis we do not consider questions on the structure of classes of matrix means with respect to the dimension of matrix algebras, we always assume that all means are operator means, i.e., with respect to infinite dimensional spaces. But depending on contexts, we sometimes use the terminology "matrix means" instead of "operator means".

In the rest of this chapter, we recall some important operator means which are considered in this thesis. For $A, B \in \mathbb{H}_n^+$ and $t \in [0, 1]$ the *t-geometric mean* $A\sharp_t B$ is defined by

$$A\sharp_t B \quad = \quad A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}.$$

It is also well-known that on the Riemannian manifold $\mathbb{H}_n^+$ the t-geometric means (for $0 \leq t \leq 1$) together form the unique geodesic curve joining A and B in $\mathbb{H}_n^+$. It is clear that the t-geometric mean is associated with the monotone function $f(x) = x^t$ for $t \in [0, 1]$.

A mean σ is called *symmetric* or *self-adjoint* if $A\sigma B = B\sigma A$ for any A and B. In this case, the representing function f of σ satisfies the identity $f(t) = tf(t^{-1})$ $(t \in (0, \infty))$. In 1979, Ando [6] highlighted that any symmetric mean σ lies between *the arithmetic mean* $A\nabla B = \dfrac{A+B}{2}$ and *the harmonic mean* $A!B := (A^{-1}\nabla B^{-1})^{-1}$, i.e.,

$$! \leq \sigma \leq \nabla.$$

Chapter 2

New types of operator convex functions and related inequalities

Being the most fundamental concept in convex analysis and optimization theory, the convexity of functions has been extensively studied in various contexts of pure and applied mathematics.

In 1906, Jensen proved that if the function f is convex on some interval $J \subset \mathbb{R}$, then for any m points $x_1, x_2, ..., x_m$ in J, we have

$$\frac{f(x_1) + f(x_2) + ... + f(x_m)}{m} \geq f\left(\frac{x_1 + x_2 + ... + x_m}{m}\right).$$

Jensen inequality is quite often used in proving many important inequalities and in determining the convexity of a function. In 1978, Breckner [19] introduced s-convex functions as a generalization of convex functions. After that, in 1985, Godunova and Levin [36] introduced a Godunova-Levin function or Q-class function and studied properties of this new type of functions as well as their applications in convex functions. Follow up this research tendency of functions, ten years later, in 1995, Dragomir et. al [28] gave a definition of P-function and some results on this class of function are shown by Pearce-CEM-Rubinov [76] and Tseng-Yang-Dragomir [82]. In 2007, Zhang and Wan [87] introduced p-convex functions. At the same year, the concept of h-convexity with a super-multiplicative function h was defined by Varošanec [84]. For more information about applications of different types of convexity the author refers the readers to [2, 37, 38, 68, 82, 88].

Let us summary all mentioned above types of convexity in the following definition.

Definition 2.0.1. Let J be an interval in $\mathbb{R}^+$ such that $[0,1] \subset J$; p and s some positive real numbers, and K ($\subset \mathbb{R}^+$) a p-convex set (i.e., $[\lambda x^p + (1-\lambda)y^p]^{1/p} \in K$ for all $x, y \in K$ and $\lambda \in [0,1]$). A function h defined on J is called super-multiplicative if $h(xy) \geq h(x)h(y)$ for any x and y in J. A function $f : K \to \mathbb{R}^+$ is said to be:

– *convex*, if $$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y);$$

– *s-convex*, if $$f(\lambda x + (1-\lambda)y) \leq [\lambda f(x)^s + (1-\lambda)f(y)^s]^{1/s};$$

– *Godunova-Levin*, if $$f(\lambda x + (1-\lambda)y) \leq \lambda^{-1} f(x) + (1-\lambda)^{-1} f(y);$$

– *P-convex (or the class $P(K)$)*, if $$f(\lambda x + (1-\lambda)y) \leq f(x) + f(y);$$

– *p-convex (or the class $PC(K)$)*, if $$f\left([\lambda x^p + (1-\lambda)y^p]^{1/p}\right) \leq \lambda f(x) + (1-\lambda)f(y);$$

– *h-convex*, if $$f(\lambda x + (1-\lambda)y) \leq h(\lambda)f(x) + h(1-\lambda)f(y),$$

for any $x, y \in K$ and $\lambda \in (0,1)$. If the inequalities are reversed, then we have the corresponding types of concave functions.

We observe that in the above definitions of convexity each type depends on the means used in the corresponding inequality. Therefore, if by changing scalar means we will obtain new classes of convex functions. Now we define a more general type of convexity which is associated with scalar means as follows.

Definition 2.0.2. Let M and N be two scalar means on $\mathbb{R}^+$. A non-negative, continuous function f is called *MN-convex* on K if for any $x, y \in K$,

$$f(M(x,y)) \leq N(f(x), f(y)). \tag{2.0.1}$$

This definition covers all types of convexity listed above.

One of the most useful means is *the binormal mean*, or *the power mean*

$$f(p) = \left(\frac{a^p + b^p}{2}\right)^{1/p}$$

which is an increasing function of p on $(-\infty, \infty)$. And it is well-known that for two positive numbers a, b,

$$\sqrt{ab} = \exp\left(\frac{1}{2}(\log a + \log b)\right) = \lim_{p \mapsto 0}\left(\frac{a^p + b^p}{2}\right)^{1/p}.$$

In [15] Bhagwat and Subramanian firstly studied the matrix version of $f(p)$. They showed that for positive semi-definite matrices A, B,

$$\lim_{p \to 0^+}\left(\frac{A^p + B^p}{2}\right)^{1/p} = \exp\left(\frac{\log A + \log B}{2}\right).$$

The mean $\exp\left(\frac{1}{2}(\log A + \log B)\right)$ is called *the log Euclidian mean* which is quite different from the geometric mean $A \sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$. The matrix power mean is a Kubo-Ando mean if and only if $p = \pm 1$.

Bhagwat and Subramanian also showed that the matrix function $f(p)$ is monotone with respect to p, on the intervals $(-\infty, -1]$ and $[1, \infty)$ but not on $(-1, 1)$. Recently, Audenaert and Hiai [12] obtained a more general result on the monotonicity of $f(p)$ which will be used in this chapter.

Theorem 2.0.1 ([12]). *Let $p, q \in \mathbb{R}$. Then the matrix inequality*

$$\left(\frac{A^p + B^p}{2}\right)^{1/p} \leq \left(\frac{A^q + B^q}{2}\right)^{1/q}$$

holds for any positive semi-definite matrices A, B if and only if p, q satisfy one of the following *conditions:*

$$\begin{cases} p = q, \\ 1 \leq p < q, \\ q < p \leq -1, \\ p \leq -1, q \geq 1, \\ 1/2 \leq p < 1 \leq q, \\ p \leq -1 < q \leq -1/2. \end{cases}$$

Now let us consider the general approach of operator convexity. Here we present what is arguably the simplest approach to these inequalities. This is accomplished by using matrix analogues of two elementary ideas from classical convexity theory: the Jensen inequality, and the construction of the perspective of a convex function.

Definition 2.0.3. Let $X = (A_1, A_2)$ with $\sigma(A_1), \sigma(A_2) \subset K$. Let M, N be arbitrary operator means (Kubo-Ando or non-Kubo-Ando means). A continuous function f is called *operator MN-convex* on K if

$$f\left(M(A_1, A_2)\right) \leq N\left(f(A_1), f(A_2)\right). \tag{2.0.2}$$

In the case

$$M(A_1, A_2) = N(A_1, A_2) = \frac{A_1 + A_2}{2}$$

if the function f satisfies inequality (2.0.2) then f is operator convex.

It is worth to remind that operator monotone functions and operator convex functions are much popular and important because of their applications in matrix analysis, matrix theory, operator theory, and especially, in quantum information theory. People characterized those functions by using several tools including matrix inequalities. One of the most famous characterizations of operator convexity due to Hansen and Pedersen [42] by Jensen's inequality.

Theorem 2.0.2 ([42]). *If f is a continuous, real-valued function on the half-open interval $K = [0, \alpha)$ $(\alpha \leq \infty)$, then the following conditions are equivalent:*

(i) *f is operator convex and $f(0) \leq 0$;*

(ii) *$f(A^*XA) \leq A^*f(X)A$ for every contraction A and every Hermitian operator X with spectrum in K;*

(iii) *$f(A^*XA + B^*YB) \leq A^*f(X)A + B^*f(Y)B$ for all operators A, B such that $A^*A + B^*B \leq I$ and for all Hermitian X, Y with spectra in K;*

(iv) *$f(PXP) \leq Pf(X)P$ for any orthogonal projection P and any self-adjoint X with spectrum in K.*

The inequality in *(ii)* is known as Davis-Choi inequality. The inequality in *(iii)* is well-known with the name *Hansen-Pedersen inequality*.

The main aim of this chapter is to define new classes of operator convex functions based on Kubo-Ando theory of operator means even for any number of matrices [77]. More precisely, we use the family of the matrix power means to define new classes of so called *operator* (r, s)-*convex functions* and *operator* (p, h)-*convex functions*. Studying their properties, we prove some well-known inequalities for them. We also provide similar to the Hansen-Pedersen characterizations for operator (p, h)-convex and operator (r, s)-convex functions.

Main results in this chapter are from the works [51] and [48].

2.1 Operator (p, h)-convex functions

The results of this section were announced in [51].

In this section, recall that let p be some positive number, J some interval in $\mathbb{R}^+$ containing the interval $[0, 1]$ (or $J \equiv [0, 1]$), and K ($\subset \mathbb{R}^+$) a p-convex subset of $\mathbb{R}^+$ (i.e., $[\alpha x^p + (1 - \alpha)y^p]^{1/p} \in K$ for all $x, y \in K$ and $\alpha \in [0, 1]$).

In [29] a more general class of non-negative functions, so-called (p, h)-convex functions is considered.

Definition 2.1.1. ([29]) Let $h : J \to \mathbb{R}^+$ be a non-zero super-multiplicative function. A non-negative function $f : K \to \mathbb{R}$ is said to be (p, h)-*convex* if

$$f\left([\alpha x^p + (1-\alpha)y^p]^{1/p}\right) \leq h(\alpha)f(x) + h(1-\alpha)f(y) \tag{2.1.3}$$

for all $x, y \in K$ and $\alpha \in [0, 1]$.

This class contains several well-known classes of functions such as non-negative convex functions, h-convex [84], r-convex functions [22, 35], Godunova-Levin functions (or Q-class functions) [36], P-class functions [28]. Recently, M. S. Moslehian and others [13, 30, 58, 70] introduced operator P-class functions and operator Q-class functions. They studied properties and proved several inequalities for these functions but characterization of such functions.

It is easy to check that the spectrum of the matrix $[\alpha A^p + (1-\alpha)B^p]^{1/p}$ belongs to K for any two matrices $A, B \in \mathbb{H}_n^+$ with spectra in K. Now we define a new class of operator (p, h)-convex functions as follows.

Definition 2.1.2. Let $h : J \to \mathbb{R}^+$ be a non-zero super-multiplicative function. A non-negative function $f : K \to \mathbb{R}$ is said to be *operator* (p, h)-*convex* (or belongs to the class $opgx(p, h, K)$) if for any $A, B \in \mathbb{H}_n^+$ with spectra in K, and for $\alpha \in [0, 1]$,

$$f\left([\alpha A^p + (1-\alpha)B^p]^{1/p}\right) \leq h(\alpha)f(A) + h(1-\alpha)f(B). \tag{2.1.4}$$

When $p = 1,\ h(\alpha) = \alpha$, we get the usual definition of operator convex functions on $\mathbb{R}^+$.

Remark 2.1.1. An operator (p, h)-convex function could be either operator monotone or operator convex. However, there are many operator (p, h)-convex functions which are neither an operator monotone function nor an operator convex function. Indeed, let $p > 0,\ f(t) = t^s$ and $h(\alpha) = \alpha$. Then the function f is operator (p, h)-convex if and only if for any positive semi-definite matrices A, B with spectra in K,

$$(\alpha A + (1-\alpha)B)^{s/p} \leq \alpha A^{s/p} + (1-\alpha)B^{s/p}.$$

The last inequality means that the function $g(t) = t^{s/p}$ is operator convex which is equivalent to the condition $s/p \in [1, 2]$, or, $s \in [p, 2p]$.

Now let us consider the following particular cases.

(i) For $s \in [p, 2p] \cap [0, 1]$, we have

$$\begin{aligned} f\left([\alpha A^p + (1-\alpha)B^p]^{1/p}\right) &= [\alpha A^p + (1-\alpha)B^p]^{s/p} \\ &\leq \alpha(A^p)^{s/p} + (1-\alpha)(B^p)^{s/p} \\ &= \alpha A^s + (1-\alpha)B^s \\ &= h(\alpha)f(A) + h(1-\alpha)f(B). \end{aligned} \tag{2.1.5}$$

The first inequality follows from the operator convexity of the function $g(t) = t^{s/p}$ since $s/p \in [1, 2]$. Therefore, the function $f(t) = t^s$ is either operator monotone or operator (p, h)-convex.

(ii) For $s \in [p, 2p] \cap [1, 2]$, the function $t^{s/p}$ is operator convex. By using similar arguments as in (2.1.5), we also get

$$\begin{aligned} f\left([\alpha A^p + (1-\alpha)B^p]^{1/p}\right) &= [\alpha A^p + (1-\alpha)B^p]^{s/p} \\ &\leq \alpha(A^p)^{s/p} + (1-\alpha)(B^p)^{s/p} \\ &= \alpha A^s + (1-\alpha)B^s \\ &= h(\alpha)f(A) + h(1-\alpha)f(B). \end{aligned}$$

Thus, the function t^s is both operator (p, h)-convex and operator convex for $s \in [1, 2]$.

(iii) Unfortunately, in the case $s \in [p, 2p] \cap (\mathbb{R} \setminus [0, 2])$, the function $f(t) = t^s$ is operator (p, h)-convex but neither operator convex nor operator monotone.

2.1.1 Some properties of operator (p, h)-convex functions

The results of this subsection are taken from [51].

In the following theorems we establish some properties of operator (p, h)-convex functions.

Theorem 2.1.1. *Let $opgx(p, h, K)$ be the class of operator (p, h)-convex functions. The following claim holds:*

(i) *If $f, g \in opgx(p, h, K)$ and $\lambda > 0$, then $f + g, \lambda f \in opgx(p, h, K)$;*

(ii) *Let h_1 and h_2 be non-negative and non-zero super-multiplicative functions defined on an interval J with $h_2 \leq h_1$ in $[0, 1]$. If $f \in opgx(p, h_2, K)$, then $f \in opgx(p, h_1, K)$;*

(iii) *Let $f \in opgx(p_2, h, K)$ such that f is operator monotone function on K. If $1 \leq p_1 \leq p_2$, then $f \in opgx(p_1, h, K)$.*

Proof. (i) Let $f, g \in opgx(p, h, K)$. Then for any self-adjoint matrices A, B with spectra in K and for $\alpha \in [0, 1]$ from the operator (p, h)-convexity of f and g,

$$f\left([\alpha A^p + (1-\alpha)B^p]^{1/p}\right) \leq h(\alpha)f(A) + h(1-\alpha)f(B)$$

and

$$g\left([\alpha A^p + (1-\alpha)B^p]^{1/p}\right) \leq h(\alpha)g(A) + h(1-\alpha)g(B).$$

Therefore,

$$\begin{aligned}(f+g)\left([\alpha A^p + (1-\alpha)B^p]^{1/p}\right) &= f\left([\alpha A^p + (1-\alpha)B^p]^{1/p}\right) + g\left([\alpha A^p + (1-\alpha)B^p]^{1/p}\right)\\ &\leq h(\alpha)f(A) + h(1-\alpha)f(B) + h(\alpha)g(A) + h(1-\alpha)g(B)\\ &= h(\alpha)(f+g)(A) + h(1-\alpha)(f+g)(B).\end{aligned}$$

and

$$(\lambda f)\left([\alpha A^p + (1-\alpha)B^p]^{1/p}\right) = \lambda f\left([\alpha A^p + (1-\alpha)B^p]^{1/p}\right)$$
$$\leq \lambda\left[h(\alpha)f(A) + h(1-\alpha)f(B)\right]$$
$$= h(\alpha)(\lambda f)(A) + h(1-\alpha)(\lambda f)(B).$$

Thus, $(f+g), \lambda f \in opgx(p, h, K)$.

(ii) Suppose that $f \in opgx(p, h_2, K)$. Since $h_2(\alpha) \leq h_1(\alpha)$ for any $\alpha \in [0,1]$, hence

$$f\left([\alpha A^p + (1-\alpha)B^p]^{1/p}\right) \leq h_2(\alpha)f(A) + h_2(1-\alpha)f(B)$$
$$\leq h_1(\alpha)f(A) + h_1(1-\alpha)f(B).$$

Thus, $f \in opgx(p, h_1, K)$.

(iii) Put $g(p) = (\alpha A^p + (1-\alpha)B^p)^{1/p}$. By Theorem 2.0.1, the function $F(p) = \left(\frac{A^p + B^p}{2}\right)^{1/p}$ is monotone increasing on $[1, +\infty)$. Then $g(p_1) \leq g(p_2)$ for $1 \leq p_1 \leq p_2$. According to the operator monotonicity of f, we have

$$f\left(g(p_1)\right) \leq f\left(g(p_2)\right) \leq h(\alpha)f(A) + h(1-\alpha)f(B).$$

Thus, $f \in opgx(p_1, h, K)$. □

The following theorem is about properties of operator (p, h)-convex functions with condition that sum of coefficients is smaller or equals to 1.

Theorem 2.1.2. *Let K be an interval in $\mathbb{R}^+$ such that $0 \in K$.*

(i) *If $f \in opgx(p, h, K)$ such that $f(0) = 0$, then*

$$f\left([\alpha A^p + \beta B^p]^{1/p}\right) \leq h(\alpha)f(A) + h(\beta)f(B) \tag{2.1.6}$$

holds for arbitrary positive definite matrices A, B with spectra in K and all $\alpha, \beta \geq 0$ such that $\alpha + \beta \leq 1$;

(ii) *Let h be a non-negative function such that $h(\alpha) < 1/2$ for some $\alpha \in (0, 1/2)$. If f is a non-negative function satisfying (2.1.6) for all matrices A, B with spectra in K and all $\alpha, \beta \geq 0$ with $\alpha + \beta \leq 1$, then $f(0) = 0$.*

Proof. (i) Let $\alpha, \beta \geq 0, \alpha + \beta = \gamma < 1$, and let a and b be numbers such that $a = \dfrac{\alpha}{\gamma}$ and $b = \dfrac{\beta}{\gamma}$. Then $a + b = 1$. Hence we have

$$\begin{aligned} f\left([\alpha A^p + \beta B^p]^{1/p}\right) &= f\left([a\gamma A^p + b\gamma B^p]^{1/p}\right) \\ &\leq h(a) f\left([\gamma A^p]^{1/p}\right) + h(b) f\left([\gamma B^p])^{1/p}\right) \\ &= h(a) f\left([\gamma A^p + (1-\gamma) O^p]^{1/p}\right) + h(b) f\left([\gamma B^p + (1-\gamma) O^p]^{1/p}\right) \\ &\leq h(a) h(\gamma) f(A) + h(b) h(\gamma) f(B) \\ &\leq h(a\gamma) f(A) + h(b\gamma) f(B) \\ &= h(\alpha) f(A) + h(\beta) f(B). \end{aligned}$$

The first and the second inequalities follows from the definition of f, the third inequality follows from the super-multiplicativity of h.

(ii) Suppose that $f(0) > 0$, then $f(O) = f(0)I$. Substituting $A = B = O$ into (2.1.6), we get

$$f(0)I = f\left([\alpha O^p + \beta O^p]^{1/p}\right) \leq h(\alpha) f(0) I + h(\beta) f(0) I. \tag{2.1.7}$$

Let $\alpha = \beta$. Dividing both sides of (2.1.7) by $f(0)$, we get a contradiction

$$2h(\alpha) \geq 1 \quad \text{for all} \quad \alpha \in (0, 1/2).$$

Thus, $f(0) = 0$.

$\square$

Corollary 2.1.1. *For $s > 0$, put $h_s(x) = x^s$ $(x > 0)$, and let $0 \in K \subset \mathbb{R}^+$. For all $f \in opgx(p, h_s, K)$, the inequality (2.1.6) holds for all $\alpha, \beta \geq 0$ with $\alpha + \beta \leq 1$ if and only if $f(0) = 0$.*

Proof. It is easy to check that $h_s(x)$ is super-multiplicative function.

If $f \in opgx(p, h_s, K)$, by Theorem 2.1.2, we just need to consider the case $\alpha, \beta \geq 0$ with $\alpha + \beta \leq 1$.

Put $\alpha + \beta = \gamma \leq 1$, and let a and b be positive numbers such that $a = \dfrac{\alpha}{\gamma}$ and $b = \dfrac{\beta}{\gamma}$. Then $a + b = 1$ and,

$$\begin{aligned}
f\left([\alpha A^p + \beta B^p]^{1/p}\right) &= f\left([a\gamma A^p + b\gamma B^p]^{1/p}\right) \\
&\leq h(a) f\left([\gamma A^p]^{1/p}\right) + h(b) f\left([\gamma B^p]^{1/p}\right) \\
&= a^s f\left([\gamma A^p]^{1/p}\right) + b^s f\left([\gamma B^p]^{1/p}\right) \\
&\leq a^s \gamma^s f(A) + a^s (1-\gamma)^s f(O) + b^s \gamma^s f(B) + b^s (1-\gamma)^s f(O) \\
&= a^s \gamma^s f(A) + b^s \gamma^s f(B) \\
&= \alpha^s f(A) + \beta^s f(B).
\end{aligned}$$

Substituting $A = B = O, \alpha = \beta = 1/k$ ($k \in \mathbb{N}, k \geq 2$) into (2.1.6), and then let k tend to the infinite, we get $f(0) \leq 0$. Since $f(0) \geq 0$ by the definition of operator (p, h)-convex functions, hence $f(0) = 0$. $\square$

2.1.2 Jensen type inequality and its applications

This subsection is taken from [51].

Recall that the weighted Jensen inequality for a convex continuous function f on an interval K is

$$f\left(\sum \lambda_i x_i\right) \leq \sum \lambda_i f(x_i)$$

for any set of positive numbers x_i in K and $\lambda_i \in [0, 1]$ such that $\sum_{i=1}^{n} \lambda_i = 1$.

In this subsection we prove a matrix Jensen inequality for operator (p, h)-convex functions.

Theorem 2.1.3. *Let h be a non-negative super-multiplicative function on J and $f \in opgx(p, h, K)$. Then for any set of k self-adjoint matrices A_i with spectra in K and any non-negative numbers*

$\alpha_i (i = 1 \cdots , k)$ *satisfying* $\sum_{i=1}^{k} \alpha_i = 1$,

$$f\left(\left[\sum_{i=1}^{k} \alpha_i A_i^p\right]^{1/p}\right) \leq \sum_{i=1}^{k} h(\alpha_i) f(A_i). \tag{2.1.8}$$

Proof. We will prove the theorem by induction on k.

When $k = 2$, inequality (2.1.8) reduces to (2.1.4).

Assume that (2.1.8) holds for any $(k-1)$ self-adjoint matrices $(k \geq 3)$ with spectra in K. We need to prove (2.1.8) for any k self-adjoint matrices with spectra in K.

$$\begin{aligned}
f\left(\left[\sum_{i=1}^{k} \alpha_i A_i^p\right]^{1/p}\right) &= f\left(\left[\sum_{i=1}^{k-1} \alpha_i A_i^p + \alpha_k A_k^p\right]^{1/p}\right) \\
&= f\left(\left[(1-\alpha_k)\left(\sum_{i=1}^{k-1} \frac{\alpha_i}{1-\alpha_k} A_i^p\right) + \alpha_k A_k^p\right]^{1/p}\right) \\
&\leq h(1-\alpha_k) f\left(\left[\sum_{i=1}^{k-1} \frac{\alpha_i}{1-\alpha_k} A_i^p\right]^{1/p}\right) + h(\alpha_k) f(A_k) \\
&\leq h(1-\alpha_k) \sum_{i=1}^{k-1} h\left(\frac{\alpha_i}{1-\alpha_k}\right) f(A_i) + h(\alpha_k) f(A_k) \\
&\leq \sum_{i=1}^{k} h(\alpha_i) f(A_i).
\end{aligned}$$

The first and the second inequalities follow from the inductive assumption while the third one follows from the super-multiplicativity of the function h.

Thus, (2.1.8) holds for any natural number k. □

Remark 2.1.2. As a consequence of Theorem 2.1.3 we obtain the Jensen inequality for some well-known functions classes.

- For $h(\alpha) = \alpha$ and $p = 1$, inequality (2.1.8) reduces to the well-known Jensen inequality for operator convex functions:

$$f\left(\sum_{i=1}^{k}\alpha_i A_i\right) \leq \sum_{i=1}^{k}\alpha_i f(A_i),$$

for $\alpha_i \in [0,1]$ and $\sum_{i=1}^{k}\alpha_i = 1$.

- For $h(\alpha) = \dfrac{1}{\alpha}$ and $p = 1$ we get the Jensen inequality for operator Q-class functions:

$$f\left(\sum_{i=1}^{k}\alpha_i A_i\right) \leq \sum_{i=1}^{k}\frac{f(A_i)}{\alpha_i},$$

for $\alpha_i \in (0,1)$ and $\sum_{i=1}^{k}\alpha_i = 1$.

- For $h(\alpha) = 1$, $p = 1$ we get the Jensen inequality for operator P-class functions:

$$f\left(\sum_{i=1}^{k}\alpha_i A_i\right) \leq \sum_{i=1}^{k} f(A_i),$$

for $\alpha_i \in [0,1]$ and $\sum_{i=1}^{k}\alpha_i = 1$.

As an application of the Jensen type inequality (2.1.8) we prove a matrix inequality for index set functions.

Let E be a finite nonempty set of positive integers and a set of positive semi-definite matrices A_i $(i \in E)$. Define an index set function $\mathcal{F}$ with respect to E and $A_{i i\in E}$ as follows:

$$\mathcal{F}(E) = h(W_E) f\left(\left[\frac{1}{W_E}\sum_{i\in E} w_i A_i^p\right]^{1/p}\right) - \sum_{i\in E} h(w_i) f(A_i), \tag{2.1.9}$$

where $W_E = \sum_{i\in E} w_i,\ w_i > 0$. The function $\mathcal{F}$ satisfies the triangle inequality in the following sense.

Theorem 2.1.4. *Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a super-multiplicative function, $f : K \to \mathbb{R}^+$ an operator (p,h)-convex. Let M and E be finite non-empty sets of positive integers such that $M \cap E = \emptyset$.*

Then for any $w_i > 0$ $(i \in M \cup E)$, and for any positive semi-definite matrices A_i $(i \in M \cup E)$ with spectra in K,

$$\mathcal{F}(M \cup E) \leq \mathcal{F}(M) + \mathcal{F}(E). \tag{2.1.10}$$

Proof. By the definition of the function $\mathcal{F}$,

$$\mathcal{F}(M \cup E) = h(W_{M\cup E}) f\left(\left[\frac{1}{W_{M\cup E}} \sum_{i\in M\cup E} w_i A_i^p\right]^{1/p}\right) - \sum_{i\in M\cup E} h(w_i) f(A_i).$$

On account of the operator (p, h)-convexity of f and the super-multiplicativity of h, we get

$$\begin{aligned}
& h(W_{M\cup E}) f\left(\left[\frac{1}{W_{M\cup E}} \sum_{i\in M\cup E} w_i A_i^p\right]^{1/p}\right) \\
&= h(W_{M\cup E}) f\left(\left[\frac{W_M}{W_{M\cup E}} \sum_{i\in M} \frac{w_i}{W_M} A_i^p + \frac{W_E}{W_{M\cup E}} \sum_{i\in E} \frac{w_i}{W_E} A_i^p\right]^{1/p}\right) \\
&\leq h(W_{M\cup E}) h\left(\frac{W_M}{W_{M\cup E}}\right) f\left(\left[\sum_{i\in M} \frac{w_i}{W_M} A_i^p\right]^{1/p}\right) \\
&+ h(W_{M\cup E}) h\left(\frac{W_E}{W_{M\cup E}}\right) f\left(\left[\sum_{i\in E} \frac{w_i}{W_E} A_i^p\right]^{1/p}\right) \\
&\leq h(W_M) f\left(\left[\frac{1}{W_M} \sum_{i\in M} w_i A_i^p\right]^{1/p}\right) + h(W_E) f\left(\left[\frac{1}{W_E} \sum_{i\in E} w_i A_i^p\right]^{1/p}\right).
\end{aligned} \tag{2.1.11}$$

Subtracting from both sides of (2.1.11) by $\sum_{i\in M\cup E} h(w_i) f(A_i)$ and using the identity

$$\sum_{i\in M\cup E} h(w_i) f(A_i) = \sum_{i\in M} h(w_i) f(A_i) + \sum_{i\in E} h(w_i) f(A_i),$$

we obtain (2.1.10). □

2.1.3 Characterizations of operator (p, h)-convex functions

The results of this subsection are taken from [51].

Recall the convex inequality for an operator convex function f on K,

$$f(\alpha A + (1-\alpha)B) \leq \alpha f(A) + (1-\alpha) f(B), \tag{2.1.12}$$

where A, B are self-adjoint matrices with spectra in K and $0 \leq \alpha \leq 1$. If we replace numbers α and $(1-\alpha)$ in the convex combination $\alpha A + (1-\alpha)B$ by matrices, we still have the following convex inequality for operator convex function [46, Theorem 4.22]:

$$f\left(CAC^* + DBD^*\right) \leq Cf(A)C^* + Df(B)D^*, \tag{2.1.13}$$

whenever Hermitian matrices C, D with spectra in K and $CC^* + DD^* = I$. This inequality is known as *the Hansen-Pedersen inequality*.

In the following theorem we prove a Hansen-Pedersen type inequality for operator (p, h)-convex functions.

Theorem 2.1.5. *Let $h : J \to \mathbb{R}^+$ be a super-multiplicative function, $f : K \to \mathbb{R}^+$ an operator (p, h)-convex function. Then for any pair of self-adjoint matrices A, B with spectra in K and for matrices C, D such that $CC^* + DD^* = I$,*

$$f\left(\left[CA^pC^* + DB^pD^*\right]^{1/p}\right) \leq 2h(1/2)\left(Cf(A)C^* + Df(B)D^*\right). \tag{2.1.14}$$

Proof. From the condition $CC^* + DD^* = I$, it implies that we can find a unitary block matrix

$$U := \begin{bmatrix} C & D \\ X & Y \end{bmatrix},$$

where the entries X and Y are chosen properly. Then

$$U^* = \begin{bmatrix} C & X^* \\ D^* & Y \end{bmatrix},$$

and

$$U \begin{bmatrix} A^p & O \\ O & B^p \end{bmatrix} U^* = \begin{bmatrix} CA^pC^* + DB^pD^* & CA^pX^* + DB^pY^* \\ XA^pC^* + YB^pD^* & XA^pX^* + YB^pY^* \end{bmatrix}.$$

For $V = \begin{bmatrix} -I & O \\ O & I \end{bmatrix}$, we have

$$\frac{1}{2}V \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} V + \frac{1}{2} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix}.$$

Identifying $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = U \begin{bmatrix} A^p & O \\ O & B^p \end{bmatrix} U^*$, we get

$$\begin{aligned} Z &:= \frac{1}{2}VU \begin{bmatrix} A^p & O \\ O & B^p \end{bmatrix} U^*V + \frac{1}{2}U \begin{bmatrix} A^p & O \\ O & B^p \end{bmatrix} U^* \\ &= \begin{bmatrix} CA^pC^* + DB^pD^* & O \\ O & XA^pX^* + YB^pY^* \end{bmatrix}. \end{aligned}$$

It implies $Z_{11} = CA^pC^* + DB^pD^*$ and $f(Z_{11}^{1/p}) = f([CA^pC^* + DB^pD^*]^{1/p})$.

On account of the (p, h)-operator convexity of f, we get

$$
\begin{aligned}
f(Z^{1/p}) &= f\left(\left[\frac{1}{2}VU\begin{bmatrix} A^p & O \\ O & B^p \end{bmatrix}U^*V + \frac{1}{2}U\begin{bmatrix} A^p & O \\ O & B^p \end{bmatrix}U^*\right]^{1/p}\right) \\
&\leq h\left(\frac{1}{2}\right)VUf\left(\begin{bmatrix} A^p & O \\ O & B^p \end{bmatrix}^{1/p}\right)U^*V + h\left(\frac{1}{2}\right)Uf\left(\begin{bmatrix} A^p & O \\ O & B^p \end{bmatrix}^{1/p}\right)U^* \\
&= 2h\left(\frac{1}{2}\right)\left(\frac{1}{2}VUf\left(\begin{bmatrix} A & O \\ O & B \end{bmatrix}\right)U^*V + \frac{1}{2}Uf\left(\begin{bmatrix} A & O \\ O & B \end{bmatrix}\right)U^*\right) \\
&= 2h\left(\frac{1}{2}\right)\begin{bmatrix} Cf(A)C^* + Df(B)D^* & O \\ O & XAX^* + YBY^* \end{bmatrix},
\end{aligned}
$$

where $$\frac{1}{2}VUU^*V + \frac{1}{2}UU^* = I.$$

Therefore,

$$
\begin{aligned}
f(Z_{11}^{1/p}) &= f([CA^pC^* + DB^pD^*]^{1/p}) \\
&\leq 2h\left(\frac{1}{2}\right)[Cf(A)C^* + Df(B)D^*].
\end{aligned}
$$

□

Tikhonov [81] showed a characterization of operator convex functions by changing roles of numbers and matrices in the Jensen inequality (2.1.12).

Lemma 2.1.1 ([81]). *For a function $f : K \to \mathbb{R}$, the following conditions are equivalent:*

(i) f is matrix convex;

(ii) for any natural number k, for any family of positive operators $\{A_i\}_{i=1}^k$ in a finite-dimensional Hilbert space $\mathbb{H}$, such that $\sum_{i=1}^k A_i = I_{\mathbb{H}}$, and arbitrary numbers $x_i \in K$, the following inequality holds,

$$f\left(\sum_{i=1}^k x_i A_i\right) \leq \sum_{i=1}^k f(x_i) A_i.$$

In the following theorem, we obtain several equivalent conditions for a function to become operator (p,h)-convex including Tikhonov's characterization. The proof of the following theorem is adapted from [42] and [81].

Theorem 2.1.6. *Let f be a non-negative function on the interval K such that $f(0) = 0$, and h a non-negative and non-zero super-multiplicative function on J satisfying $2h(1/2) \leq \alpha^{-1}h(\alpha)$ $(\alpha \in (0,1))$. Then the following statements are equivalent:*

(i) *f is an operator (p,h)-convex function;*

(ii) *for any contraction V ($||V|| \leq 1$) and self-adjoint matrix A with spectrum in K,*

$$f\left([V^*A^pV]^{1/p}\right) \leq 2h(1/2)V^*f(A)V;$$

(iii) *for any orthogonal projection Q and any self-adjoint matrix A with $\sigma(A) \subset K$,*

$$f\left([QA^pQ]^{1/p}\right) \leq 2h(1/2)Qf(A)Q;$$

(iv) *for any natural number k, for any families of positive operators $\{A_i\}_{i=1}^k$ in a finite dimensional Hilbert space $\mathbb{H}$ satisfying $\sum_{i=1}^k \alpha_i A_i = I_{\mathbb{H}}$ (the identity operator in $\mathbb{H}$) and for arbitrary numbers $x_i \in K$,*

$$f\left(\left[\sum_{i=1}^k \alpha_i x_i^p A_i\right]^{1/p}\right) \leq \sum_{i=1}^k h(\alpha_i) f(x_i) A_i. \tag{2.1.15}$$

Proof. Every orthogonal projection with norm of 1 is a contraction. Therefore, the implication *(ii)* $\Rightarrow$ *(iii)* is obvious.

Let us prove the implication *(i)* $\Rightarrow$ *(ii)*. Suppose that $f \in opgx(p,h,K)$. Then by Theorem 2.1.5, for any matrices $C, D \in \mathbb{M}_n$ such that $CC^* + DD^* = I$ we have

$$f\left([CA^pC^* + DB^pD^*]^{1/p}\right) \leq 2h(1/2)\left(Cf(A)C^* + Df(B)D^*\right).$$

Since $||V|| \leq 1$, we can choose W such that $VV^* + WW^* = I$. If we choose $B = O$, then we get that $f(B) = f(O) = O$. Therefore,

$$\begin{aligned} f\left([V^*A^pV]^{1/p}\right) &= f((V^*A^pV + W^*B^pW)^{1/p}) \\ &\leq 2h(1/2)\left(V^*f(A)V + W^*f(B)W\right) \\ &\leq 2h(1/2)(V^*f(A)V). \end{aligned}$$

(iii) $\Rightarrow$ *(i)*. Let A and B be self-adjoint matrices with spectra in K and $0 < \alpha < 1$. Define

$$C := \begin{bmatrix} A & O \\ O & B \end{bmatrix},\ U := \begin{bmatrix} \sqrt{\alpha}I & -\sqrt{1-\alpha}I \\ \sqrt{1-\alpha}I & \sqrt{\alpha}I \end{bmatrix},\ Q := \begin{bmatrix} I & O \\ O & O \end{bmatrix}.$$

Then $C = C^*$ with $\sigma(C) \subset K$, U is a unitary and Q is an orthogonal projection. Furthermore,

$$U^*C^pU = \begin{bmatrix} \alpha A^p + (1-\alpha)B^p & -\sqrt{\alpha-\alpha^2}A^p + \sqrt{\alpha-\alpha^2}B^p \\ -\sqrt{\alpha-\alpha^2}A^p + \sqrt{\alpha-\alpha^2}B^p & (1-\alpha)A^p + \alpha B^p \end{bmatrix}$$

is a self-adjoint matrix. Since

$$QU^*C^pUQ = \begin{bmatrix} \alpha A^p + (1-\alpha)B^p & O \\ O & O \end{bmatrix},$$

and $||UQ|| \leq 1$, we get

$$\begin{aligned} \begin{bmatrix} f\left([\alpha A^p + (1-\alpha)B^p]^{1/p}\right) & O \\ O & O \end{bmatrix} &= f\left((QU^*C^pUQ)^{1/p}\right) \\ &\leq 2h(1/2)QU^*f(C)UQ \\ &= 2h(1/2)\begin{bmatrix} \alpha f(A) + (1-\alpha)f(B) & O \\ O & O \end{bmatrix}. \end{aligned}$$

According to the property of h, the last inequality implies

$$f\left([\alpha A^p+(1-\alpha)B^p]^{1/p}\right) \le 2h(1/2)\left[\alpha f(A)+(1-\alpha)f(B)\right]$$
$$\le h(\alpha)f(A)+h(1-\alpha)f(B).$$

(iv) $\Rightarrow$ *(i)*. Let X, Y be two arbitrary self-adjoint operators on $\mathbb{H}$ with spectra in K, and $\alpha \in (0,1)$. Let

$$X=\sum_{i=1}^{n}\lambda_i P_i, \quad Y=\sum_{j=1}^{n}\mu_j Q_j,$$

be the spectral decompositions of X and Y. Then it follows that

$$\alpha\sum_{i=1}^{n}P_i+(1-\alpha)\sum_{j=1}^{n}Q_j=I_{\mathbb{H}},$$

where $I_{\mathbb{H}}$ is the identity operator on $\mathbb{H}$. On account of (2.1.15),

$$\begin{aligned} f\left([\alpha X^p+(1-\alpha)Y^p]^{1/p}\right) &= f\left(\left[\sum_{i=1}^{n}\alpha\lambda_i^p P_i+\sum_{j=1}^{n}(1-\alpha)\mu_i^p Q_j\right]^{1/p}\right), \\ &\le \sum_{i=1}^{n}h(\alpha)f(\lambda_i)P_i+\sum_{j=1}^{n}h(1-\alpha)f(\mu_j)Q_j, \\ &= h(\alpha)\sum_{i=1}^{n}f(\lambda_i)P_i+h(1-\alpha)\sum_{j=1}^{n}f(\mu_j)Q_j, \\ &= h(\alpha)f(X)+h(1-\alpha)f(Y). \end{aligned}$$

(i) $\Rightarrow$ *(iv)*. By applying the proof of Lemma 2.1.1 *(ii)*, by Naimark's theorem [71], there exists a Hilbert space $\mathcal{H}$ containing $\mathbb{H}$ and a family of mutually orthogonal projections P_i in $\mathcal{H}$ such that $\sum_{i=1}^{k}P_i=I_{\mathcal{H}}$ and $\alpha_i A_i = PP_iP|_{\mathbb{H}}(i=1,2,...,k)$, where P is the projection from $\mathcal{H}$

onto $\mathbb{H}$. Then we have

$$\begin{aligned}
f\left(\left[\sum_{i=1}^{k}\alpha_i x_i^p A_i\right]^{1/p}\right) &= f\left(\left[\sum_{i=1}^{k} x_i^p PP_iP|_{\mathbb{H}}\right]^{1/p}\right),\\
&= f\left(\left[P\left(\sum_{i=1}^{k} x_i^p P_i\right)P|_{\mathbb{H}}\right]^{1/p}\right),\\
&\leq 2h\left(\frac{1}{2}\right)Pf\left(\left[\sum_{i=1}^{k} x_i^p P_i\right]^{1/p}\right)P|_{\mathbb{H}},\\
&= 2h\left(\frac{1}{2}\right)P\left(\sum_{i=1}^{k} f(x_i)P_i\right)P|_{\mathbb{H}},\\
&= 2h\left(\frac{1}{2}\right)\sum_{i=1}^{k} f(x_i)PP_iP|_{\mathbb{H}},\\
&= 2h\left(\frac{1}{2}\right)\sum_{i=1}^{k} \alpha_i f(x_i)A_i,\\
&\leq \sum_{i=1}^{k} h(\alpha_i)f(x_i)A_i.
\end{aligned}$$

□

Remark 2.1.3. Here we give an example for the function h which is different from the identify function and satisfies the conditions in Theorem 2.1.6. It is easy to check that for the function $h(x) = x^3 - x^2 + x$ and for any $x, y \in [0, 1]$,

$$h(xy) - h(x)h(y) = xy(x + y)(1 - x)(1 - y) \geq 0.$$

Therefore, h is super-multiplicative on $[0, 1]$. At the same time, the function $h(x)/x = x^2 - x + 1$ attains minimum at $x = 1/2$, and hence $2h(1/2) \leq h(x)/x$ for any $x \in (0, 1)$.

In the following corollary we obtain a relation between operator (p, h)-convex functions and operator monotone functions on $\mathbb{R}^+$.

Corollary 2.1.2. *Let f be operator $(1, h)$-convex function on $\mathbb{R}^+$ such that $f(0) = 0$. Then for any positive definite matrices $A \le B$,*

$$A^{-1} f(A) \le 2h(1/2) B^{-1} f(B).$$

In the case when $2h(1/2) \le 1$, the function $t^{-1} f(t)$ is operator monotone on $(0, \infty)$, and hence the function $f(t)$ is operator convex.

Proof. Let $0 \le A \le B$. Then we can find C such that $A^{1/2} = CB^{1/2}$, and hence $A = CBC^*$. Therefore,

$$\begin{aligned}A^{-1} f(A) &= B^{-1/2} C^{-1} f(CBC^*)(C^*)^{-1} B^{-1/2} \\ &\le 2h(1/2) B^{-1/2} C^{-1} C f(B) C^* (C^*)^{-1} B^{-1/2} \\ &= 2h(1/2) B^{-1} f(B).\end{aligned}$$

In the case when $2h(1/2) \le 1$, from the above inequality we get

$$A^{-1} f(A) \le B^{-1} f(B),$$

that means, the function $t^{-1} f(t)$ is operator monotone, and as a consequence of that, the function $f(t)$ is operator convex by [42]. $\square$

Remark 2.1.4. It is easy to check that the function $h(x) = (x^3 - x^2 + x)/2$ is super-multiplicative and satisfies the conditions in Theorem 2.1.6 and Corollary 2.1.2.

In next section, let r, s be positive numbers. We define a new class of operator (r, s)-convex functions with respect to the matrix power mean, and study their properties. We also provide a series of equivalent conditions for a function to be operator (r, s)-convex.

2.2 Operator (r, s)-convex functions

The results of this section are taken from [48].

Let r, s be positive numbers. For $X = (A_1, A_2)$ of Hermitian matrices with spectra in K and a function f. We denote $f(X) = (f(A_1), f(A_2))$. For a pair of positive numbers $W = (\omega_1, \omega_2)$. We set $W_2 := \omega_1 + \omega_2$ and define the weighted matrix r-power mean $M^{[r]}(X, W)$ by

$$M^{[r]}(X, W) := \left[\frac{1}{W}(\omega_1 A_1^r + \omega_2 A_2^r)\right]^{1/r}.$$

Definition 2.2.1. Let K be a r-convex subset of $\mathbb{R}^+$. A continuous function $f : K \to (0, \infty)$ is said to be *operator* (r, s)*-convex* if

$$f(M^{[r]}(X, W)) \le M^{[s]}(f(X), W) \tag{2.2.16}$$

If the inequality (2.2.16) is reversed, f is *operator* (r, s)*-concave*.

Remark 2.2.1. The reader may notice a similarity between this notion with the notion of (p, h)-convex functions introduced in the previous section. However there should be no confusion as h is a non-constant function (being a super-multiplicative function).

Motivated by two specific cases of operator (p, h)-convexity and operator (r, s)-convexity, we intend to consider a general definition for this type of operator convexity as follows: Let p, q be positive real numbers, h be super-multiplicative non-negative real valued function. A function f is called operator (p, h, q)-convex if

$$f\left([\alpha A^p + (1-\alpha)B^p]^{1/p}\right) \le [h(\alpha)f(A)^q + h(1-\alpha)f(B)^q]^{1/q}, \ \alpha \in [0, 1].$$

If $q = 1$ then we get the class of operator $(p, h, 1)$-convex or called operator (p, h)-convex, and if $h \equiv id$ is identity function, we get the class of operator (p, id, q)-convex functions, or called as operator (r, s)-convex functions. In the future, we intend to continue to investigate this general class of operator functions for some different cases.

Notice that the class of operator convex functions is very important in matrix analysis and quantum information theory. The class of operator log-convex functions was studied by Hiai and Ando [11] and got fully characterized as operator decreasing functions.

Based on the results in theorem 2.0.1 of Hiai and Audenaert [12], one can describe all power functions which are operator (r, s)-convex on $\mathbb{R}^+$. Indeed, let r and s be two positive numbers, $f(x) = x^\alpha$ $(\alpha \in \mathbb{R})$. For $s \geq 1$ and $\dfrac{\alpha s}{r} \in [1, 2]$, the function $t^{\frac{\alpha s}{r}}$ is operator convex, and $t^{1/s}$ is operator monotone. Then,

$$\begin{aligned} f(M^{[r]}) = (M^{[r]})^\alpha &= \left\{ \left[\frac{1}{W}(\omega_1 A_1^r + \omega_2 A_2^r) \right]^{1/r} \right\}^\alpha \\ &= \left\{ \left[\frac{1}{W}(\omega_1 A_1^r + \omega_2 A_2^r) \right]^{\alpha s/r} \right\}^{1/s} \\ &\leq \left[\frac{1}{W}(\omega_1 A_1^{\alpha s} + \omega_2 A_2^{\alpha s}) \right]^{1/s} \\ &= \left(\frac{1}{W} \left[\omega_1 f(A_1)^s + \omega_2 f(A_2)^s \right] \right)^{1/s} \\ &= M^{[s]} \left(f(X), W \right). \end{aligned}$$

Thus, f is an operator (r, s)-convex function.

For simplicity, we write $f(A)^s$ instead of $\{f(A)\}^s$.

If the function $t^{1/s}$ is operator convex, by using the similar arguments we can see that f is operator (r, s)-convex.

Another example of operator (r, s)-convex functions are given by F. Hiai as follows: For $s, r > 0$ and for a function $f : [0, \infty) \to \mathbb{R}$, we denote $f_{s,r}(x) = [f(x^{1/r})]^s$. Then, by replacing A^r, B^r with A, B, the inequality (2.2.16) is rewritten as

$$\left[f_{s,r} \left(\frac{A + B}{2} \right) \right]^{1/s} \leq \left[\frac{f_{s,r}(A) + f_{s,r}(B)}{2} \right]^{1/s}. \tag{2.2.17}$$

Therefore, for $s \geq 1$, a sufficient condition for (2.2.17) to hold is that $f_{s,r}$ is operator convex.

For example, if $f_{s,r}(x) = x\log x$, then $f(x) = r^{1/s}(x^r \log x)^{1/s}$. Hence (2.2.17) holds for $f(x) = (x^r \log x)^{1/s}$ with $r > 0$ and $s \geq 1$. On the other hand, if $0 < s \leq 1$, then the operator convexity of $f_{s,r}$ is a necessary condition for (2.2.17) to hold. Also, for any $s > 0$, the numerical convexity of $f_{s,r}$ is a necessary condition.

In this section, we study some basic properties of operator (r, s)-convex functions. We also prove the Jensen, Hansen-Pedersen and Rado type inequalities for them. Some equivalent conditions for a function f to be operator (r, s)-convex are also provided.

We also obtain some properties of operator (r, s)-convex functions in the following propositions.

Proposition 2.2.1. *Let f be a continuous function on K and $1 < s \leq s'$. The following assertions hold.*

(i) If f is operator (r, s)-convex then f is also operator (r, s')-convex;

(ii) If f is operator (r, s')-concave then f is also operator (r, s)-concave.

Proof. Let f be operator (r, s)-convex and $s \leq s'$. Then the function $t^{s/s'}$ is operator concave. We have

$$\begin{aligned}
M^{[s']}\left(f(X), W\right) &= \left[\frac{\omega_1}{W} f(A_1)^{s'} + \frac{\omega_2}{W} f(A_2)^{s'}\right]^{1/s'} \\
&= \left(\left[\frac{\omega_1}{W} f(A_1)^{s'} + \frac{\omega_2}{W} f(A_2)^{s'}\right]^{s/s'}\right)^{1/s} \\
&\geq \left[\frac{\omega_1}{W} f(A_1)^{s} + \frac{\omega_2}{W} f(A_2)^{s}\right]^{1/s} \\
&= M^{[s]}(f(X), W) \\
&\geq f(M^{[r]}(X, W)).
\end{aligned}$$

Thus, f is also operator (r, s')-convex. The second property can be proved similarly. $\square$

Proposition 2.2.2. *Let f, g be continuous on K and $r, s > 0$.*

(i) If f is operator (r, s)-convex and $\alpha > 0$, then so is αf,

(ii) If f, g are operator (r,s)-convex and $s \in [1/2, 1]$, then $(f+g)$ is r-convex.

Proof. (i) trivially follows from the definition of f. We provide a proof of *(ii)*. Let f, g be operator (r,s)-convex functions and $s \in [1/2, 1]$. Then the function $t^{1/s}$ is operator convex.

$$\begin{aligned}
(f+g)\left(M^{[r]}(X,W)\right) &= f\left(M^{[r]}(X,W)\right) + g\left(M^{[r]}(X,W)\right) \\
&= f\left(\left[\frac{\omega_1}{W}A_1^r + \frac{\omega_2}{W}A_2^r\right]^{1/r}\right) + g\left(\left[\frac{\omega_1}{W}A_1^r + \frac{\omega_2}{W}A_2^r\right]^{1/r}\right) \\
&\le \left(\left[\frac{\omega_1}{W}f(A_1)^s + \frac{\omega_2}{W}f(A_2)^s\right]\right)^{1/s} + \left(\left[\frac{\omega_1}{W}g(A_1)^s + \frac{\omega_2}{W}g(A_2)^s\right]\right)^{1/s} \\
&\le \frac{\omega_1}{W}f(A_1) + \frac{\omega_2}{W}f(A_2) + \frac{\omega_1}{W}g(A_1) + \frac{\omega_2}{W}g(A_2) \\
&= \frac{\omega_1}{W}(f+g)(A_1) + \frac{\omega_2}{W}(f+g)(A_2).
\end{aligned}$$

The first inequality follows from the definition of f, the second one follows from the operator convexity of $t^{1/s}$. Thus, $(f+g)$ is operator r-convex. $\square$

Remark 2.2.2. If s does not belong to $[1/2, 1]$, the function $(f+g)$ may not be operator r-convex even f and g are operator (r,s)-convex. Indeed, for $s = 2$ the function $t^{1/2}$ is operator concave. It is easy to see that $f(x) = x^{\frac{2r}{3}}$ and $g(x) = x^{\frac{5r}{6}}$ are operator (r,s)-convex. At the same time, we have

$$\begin{aligned}
(f+g)\left(\left[\frac{\omega_1}{W}A_1^r + \frac{\omega_2}{W}A_2^r\right]^{1/r}\right) &= f\left(\left[\frac{\omega_1}{W}A_1^r + \frac{\omega_2}{W}A_2^r\right]^{1/r}\right) + g\left(\left[\frac{\omega_1}{W}A_1^r + \frac{\omega_2}{W}A_2^r\right]^{1/r}\right) \\
&= \left(\frac{\omega_1}{W}A_1^r + \frac{\omega_2}{W}A_2^r\right)^{2/3} + \left(\frac{\omega_1}{W}A_1^r + \frac{\omega_2}{W}A_2^r\right)^{5/6} \\
&\ge \frac{\omega_1}{W}A_1^{2r/3} + \frac{\omega_2}{W}A_2^{2r/3} + \frac{\omega_1}{W}A_1^{5r/6} + \frac{\omega_2}{W}A_2^{5r/6} \\
&= \frac{\omega_1}{W}(f+g)(A_1) + \frac{\omega_1}{W}(f+g)(A_2).
\end{aligned}$$

Therefore, $(f+g)$ is operator r-concave.

2.2.1 Jensen and Rado type inequalities

The results of this section are taken from [48].

In the following theorem we prove a Jensen type inequality for operator (r,s)-convexity. Let $X=(A_1,...,A_m)$ be Hermitian matrices with spectra in K and $W=(\omega_1,...,\omega_m)$ be positive numbers. Set $W_m=\omega_1+...+\omega_m$. The weighted matrix r-power mean $M_m^{[r]}(X,W)$ is defined by

$$M_m^{[r]}(X,W) := \left(\frac{1}{W_m}\sum_{i=1}^{m}\omega_i A_i^r\right)^{1/r}.$$

Theorem 2.2.1. *Let r,s be arbitrary positive numbers such that $s\geq 1$, and m be a natural number. If f is operator (r,s)-convex then*

$$f(M_m^{[r]}(X,W)) \leq M_m^{[s]}(f(X),W). \tag{2.2.18}$$

where $X=(A_1,...,A_m)$ and $W_m=\omega_1+...+\omega_m$.

When f is operator (r,s)-concave, the inequality (2.2.18) is reversed.

Proof. We prove the theorem by mathematical induction.

With $m=2$, the inequality holds by the Definition 2.2.1. Suppose that (2.2.18) holds for $(m-1)$, i.e.,

$$f(M_{m-1}^{[r]}(X,W)) \leq M_{m-1}^{[s]}(f(X),W).$$

We prove (2.2.18) for m. We have

$$\begin{aligned}
f\left(M_m^{[r]}(X,W)\right) &= f\left(\left[\frac{1}{W_m}\sum_{i=1}^{m}\omega_i A_i^r\right]^{1/r}\right)\\
&= f\left(\left[\frac{W_{m-1}}{W_m}\sum_{i=1}^{m-1}\frac{\omega_i}{W_{m-1}}A_i^r+\frac{\omega_m}{W_m}A_m^r\right]^{1/r}\right)\\
&\leq \left[\frac{W_{m-1}}{W_m}f\left(\left[\sum_{i=1}^{m-1}\frac{\omega_i}{W_{m-1}}A_i^r\right]^{1/r}\right)^s+\frac{\omega_m}{W_m}f(A_m)^s\right]^{1/s}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\frac{W_{m-1}}{W_m} \left[\sum_{i=1}^{m-1} \frac{\omega_i}{W_{m-1}} f(A_i)^s \right] + \frac{\omega_m}{W_m} f(A_m)^s \right)^{1/s} \\
&= \left[\sum_{i=1}^{m} \frac{\omega_i}{W_m} f(A_i)^s \right]^{1/s} \\
&= M_m^{[s]} \left(f(X), W \right).
\end{aligned}
$$

The last inequality follows from the inductive assumption and the operator monotonicity of the function $x^{1/s}$. □

Now, for a_i $(i = 1, ..., m)$ to be positive number, $a = (a_1, a_2, ..., a_m)$. Let us denote the arithmetic mean $A_k(a)$ and the geometric mean $G_k(a)$ as follows:

$$
A_k(a) = \frac{1}{k} \sum_{i=1}^{k} a_i, \quad G_k(a) = \sqrt[k]{a_1 a_2 ... a_k},
$$

where $k \in \{1, 2, ..., m\}$. Let f be convex function. The Rado inequality is known in the literature as follows:

$$
m \left[\frac{\sum_{i=1}^{m} f(x_i)}{m} - f\left(A_m(x_i)\right) \right] \geq (m-1) \left[\frac{\sum_{i=1}^{m-1} f(x_i)}{m-1} - f\left(A_{m-1}(x_i)\right) \right]
$$

Also, we proves a Rado type inequality for operator (r, s)-convex functions.

Theorem 2.2.2. *Let r and s be two positive numbers and f a continuous function on K. For $m \in \mathbb{N}$, $X = (A_1, ..., A_m)$ and $W = (\omega_1, ..., \omega_m)$, we denote*

$$
a_m = W_m \left(M_m^{[s]} \left[f(X), W \right]^s - f\left(M_m^{[r]}[X, W] \right)^s \right). \tag{2.2.19}
$$

Then, the following assertions hold:

(i) If f is operator (r, s)-convex then $\{a_m\}_{m=1}^{\infty}$ is an increasing monotone sequence;

(ii) If f is operator (r, s)-concave then $\{a_m\}_{m=1}^{\infty}$ is a decreasing monotone sequence.

Proof. We have

$$\begin{aligned}
f\left[M_m^{[r]}(X,W)\right]^s &= f\left(\left[\frac{1}{W_m}\sum_{i=1}^{m}\omega_i A_i^r\right]^{1/r}\right)^s \\
&= f\left(\left[\frac{W_{m-1}}{W_m}\sum_{i=1}^{m-1}\frac{\omega_i}{W_{m-1}}A_i^r + \frac{m}{W_m}A_m^r\right]^{1/r}\right)^s \\
&\le \frac{W_{m-1}}{W_m} f\left(\left[\sum_{i=1}^{m-1}\frac{\omega_i}{W_{m-1}}A_i^r\right]^{1/r}\right)^s + \frac{\omega_m}{W_m} f(A_m)^s.
\end{aligned}$$

Consequently,

$$W_m f\left(M_m^{[r]}(X,W)\right)^s \le \omega_m f(A_m)^s + W_{m-1} f\left(M_{m-1}^{[r]}(X,W)\right)^s.$$

Therefore,

$$\begin{aligned}
a_m &= W_m\left(\frac{1}{W_m}\sum_{i=1}^{m}\omega_i f(A_i)^s - f\left(M_m^{[r]}(X,W)\right)^s\right) \\
&= \sum_{i=1}^{m}\omega_i f(A_i)^s - W_m f\left(M_m^{[r]}(X,W)\right)^s \\
&\ge \sum_{i=1}^{m}\omega_i f(A_i)^s - \omega_m f(A_m)^s - W_{m-1} f\left(M_{m-1}^{[r]}(X,W)\right)^s \\
&= \sum_{i=1}^{m-1}\omega_i f(A_i)^s - W_{m-1} f\left(M_{m-1}^{[r]}(X,W)\right)^s \\
&= W_{m-1}\left[M_{m-1}^{[s]}\left(f(X,W)\right)^s - f\left(M_{m-1}^{[r]}(X,W)\right)^s\right] = a_{m-1}.
\end{aligned}$$

$\square$

2.2.2 Some equivalent conditions to operator (r, s)-convexity

The results of this section are taken from [48].

We replace the numbers $\frac{w_1}{W}$ and $\frac{w_2}{W}$ in the combination $\frac{w_1}{W} A_1^r + \frac{w_2}{W} A_2^r$ by matrices. We get the following result.

Theorem 2.2.3. *Let $f : K \to \mathbb{R}^+$ be an operator (r, s)-convex function. Then for any pair of positive definite A, B with spectra in K and for matrices C, D such that $CC^* + DD^* = I$,*

$$f((CA^rC^* + DB^rD^*)^{1/r}) \leq (Cf(A)^sC^* + Df(B)^sD^*)^{1/s}. \tag{2.2.20}$$

Proof. Proving similarly as in Theorem 2.1.5, we also find a unitary block matrix

$$U := \begin{bmatrix} C & D \\ X & Y \end{bmatrix},$$

and define the matrix

$$Z :== \begin{bmatrix} CA^rC^* + DB^rD^* & O \\ O & XA^rX^* + YB^rY^* \end{bmatrix}$$

is diagonal, where $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = U \begin{bmatrix} A^r & O \\ O & B^r \end{bmatrix} U^*$ and $V = \begin{bmatrix} -I & O \\ O & I \end{bmatrix}$.

It implies $Z_{11} = CA^rC^* + DB^rD^*$ and $f(Z_{11}^{1/r}) = f((CA^rC^* + DB^rD^*)^{1/r})$.

On account of the (r,s)-operator convexity of f, we have

$$\begin{aligned}
f(Z^{1/r}) &= f\left(\left(\frac{1}{2}VU\begin{bmatrix} A^r & O \\ O & B^r \end{bmatrix}U^*V + \frac{1}{2}U\begin{bmatrix} A^r & O \\ O & B^r \end{bmatrix}U^*\right)^{1/r}\right) \\
&\leq \left(\frac{1}{2}f\left[\left(VU\begin{bmatrix} A^r & O \\ O & B^r \end{bmatrix}U^*V\right)^{1/r}\right]^s + \frac{1}{2}f\left[\left(U\begin{bmatrix} A^r & O \\ O & B^r \end{bmatrix}U^*\right)^{1/r}\right]^s\right)^{\frac{1}{s}} \\
&= \left(\frac{1}{2}VUf\left(\begin{bmatrix} A & O \\ O & B \end{bmatrix}\right)^s U^*V + \frac{1}{2}Uf\left(\begin{bmatrix} A & O \\ O & B \end{bmatrix}\right)^s U^*\right)^{\frac{1}{s}} \\
&= \begin{bmatrix} Cf(A)^sC^* + Df(B)^sD^* & O \\ O & Xf(A)^sX^* + Yf(B)^sY^* \end{bmatrix}^{\frac{1}{s}},
\end{aligned}$$

where $\frac{1}{2}VUU^*V + \frac{1}{2}UU^* = I$. Therefore,

$$\begin{aligned}
f(Z_{11}^{1/r}) &= f([CA^rC^* + DB^rD^*]^{1/r}) \\
&\leq [Cf(A)^sC^* + Df(B)^sD^*]^{\frac{1}{s}}.
\end{aligned}$$

□

In the following theorem, we obtain several equivalent conditions for a function to be operator (r,s)-convex.

Theorem 2.2.4. *Let f be a non-negative function on the interval K such that $f(0) = 0$. Then the following statements are equivalent:*

(i) f is an operator (r,s)-convex function;

(ii) for any contraction V $(||V|| \leq 1)$ and for any positive semi-definite matrix A with spectrum in K,

$$f\left([V^*A^rV]^{1/r}\right) \leq (V^*f(A)^sV)^{1/s};$$

(iii) for any orthogonal projection Q and for any positive semi-definite matrix A with $\sigma(A) \subset K$,

$$f\left([QA^rQ]^{1/r}\right) \le (Qf(A)^sQ)^{1/s};$$

(iv) for any natural number k and for any families of positive operators $\{A_i\}_{i=1}^k$ in a finite dimensional Hilbert space $\mathbb{H}$ such that $\sum_{i=1}^k \alpha_i A_i = I_{\mathbb{H}}$ (the identity operator in $\mathbb{H}$) and for arbitrary numbers $x_i \in K$,

$$f\left(\left[\sum_{i=1}^k \alpha_i x_i^r A_i\right]^{1/r}\right) \le \left(\sum_{i=1}^k \alpha_i f(x_i)^s A_i\right)^{1/s}. \tag{2.2.21}$$

Proof. Let us prove the implication *(i)* $\Rightarrow$ *(ii)*.

Suppose that f is an operator (r,s)-convex function. Then by Theorem 2.2.3 we have

$$f(CA^rC^* + DB^rD^*)^{1/r} \le [Cf(A^s)C^* + Df(B^s)D^*]^{1/s}$$

whenever $CC^* + DD^* = I$. Since $||V|| \le 1$, we can choose W such that $VV^* + WW^* = I$. Choosing $B = O$, we have that $f(B) = f(O) = f(0)O = O$. Hence,

$$\begin{aligned} f\left((V^*A^rV)^{1/r}\right) &\le f\left((V^*A^rV + W^*B^rW)^{1/r}\right) \\ &\le [V^*f(A)^sV + W^*f(B)^sW]^{1/s} \\ &= [V^*f(A)^sV]^{1/s}. \end{aligned}$$

The implication *(ii)* $\Rightarrow$ *(iii)* is obvious.

(iii) $\Rightarrow$ *(i)*. Let A and B be self-adjoint matrices with spectra in K and $0 < \lambda < 1$. Define

$$C := \begin{bmatrix} A & O \\ O & B \end{bmatrix}, U := \begin{bmatrix} \sqrt{\lambda}I & -\sqrt{1-\lambda}I \\ \sqrt{1-\lambda} & \sqrt{\lambda}I \end{bmatrix}, Q := \begin{bmatrix} I & O \\ O & O \end{bmatrix}.$$

Then $C = C^*$ with $\sigma(C) \subset K$, and U is a unitary, Q is an orthogonal projection and

$$U^*C^rU = \begin{bmatrix} \lambda A^r + (1-\lambda)B^r & -\sqrt{\lambda-\lambda^2}A^r + \sqrt{\lambda-\lambda^2}B^r \\ -\sqrt{\lambda-\lambda^2}A^r + \sqrt{\lambda-\lambda^2}B^r & (1-\lambda)A^r + \lambda B^r \end{bmatrix}$$

is Hermitian. Since

$$QU^*C^rUQ = \begin{bmatrix} \lambda A^r + (1-\lambda)B^r & O \\ O & O \end{bmatrix}$$

and $||UP|| \leq 1$, hence

$$\begin{aligned} f\left(\begin{bmatrix} \lambda A^r + (1-\lambda)B^r & O \\ O & O \end{bmatrix}\right)^{1/r} &= f\left((QU^*C^rUQ)^{1/r}\right) \\ &\leq [QU^*f(C)^sUQ]^{1/s} \\ &= \begin{bmatrix} [\lambda f(A)^s + (1-\lambda)f(B)^s]^{1/s} & O \\ O & O \end{bmatrix}. \end{aligned}$$

Therefore, $f\left(\lambda A^r + (1-\lambda)B^r\right)^{1/r} \leq [\lambda f(A)^s + (1-\lambda)f(B)^s]^{1/s}$.

(iv) $\Rightarrow$ *(i)*. Let X, Y be two arbitrary self-adjoint operators on $\mathbb{H}$ with spectra in K, and $\alpha \in (0,1)$. Let $X = \sum_{i=1}^{n} \lambda_i P_i$ and $Y = \sum_{j=1}^{n} \mu_j Q_j$ be the spectral decompositions of X and Y, respectively. Then we have

$$\alpha \sum_{i=1}^{n} P_i + (1-\alpha) \sum_{j=1}^{n} Q_j = I_{\mathbb{H}}.$$

On account of (2.2.21), we have

$$f\left([\alpha A^r + (1-\alpha)B^r]^{1/r}\right) = f\left(\left[\alpha \sum_{i=1}^{n} \lambda_i^r P_i + (1-\alpha) \sum_{j=1}^{n} \mu_j^r Q_j\right]^{1/r}\right)$$

$$\leq \left(\alpha f\left[\left(\sum_{i=1}^{n}\lambda^{r}\right)^{1/r}\right]^{s}P_i+(1-\alpha)f\left[\left(\sum_{j=1}^{n}\mu_j^{r}\right)^{1/r}\right]^{s}Q_j\right)^{1/s}$$

$$=\left[\alpha f(\sum_{i=1}^{n}\lambda_iP_i)^{s}+(1-\alpha)f(\sum_{j=1}^{n}\mu_jQ_j)^{s}\right]^{1/s}$$

$$\leq\left[\alpha f(A)^{s}+(1-\alpha)f(B)^{s}\right]^{1/s}.$$

(i) $\Rightarrow$ (iv). Naimark's theorem [71] states that there exitsts a Hilbert space $\mathcal{H}$ containing $\mathbb{H}$ and a family of mutually orthogonal projections P_i in $\mathcal{H}$ such that $\sum_{i=1}^{k}P_i = I_{\mathcal{H}}$ and $\alpha_iA_i = PP_iP|\mathbb{H}$ $(i = 1, 2, \cdots, k)$, where P is the projection from $\mathcal{H}$ onto $\mathbb{H}$ and $I_{\mathcal{H}}$ is the identity operator in $\mathcal{H}$. Then,

$$\begin{aligned}
f\left(\left[\sum_{i=1}^{k}\alpha_ix_i^{r}A_i\right]^{1/r}\right) &= f\left(\left[\sum_{i=1}^{k}x_i^{r}PP_iP|\mathbb{H}\right]^{1/r}\right)\\
&= f\left(\left[P\sum_{i=1}^{k}x_i^{r}P_iP|\mathbb{H}\right]^{1/r}\right)\\
&\leq \left(Pf\left(\left[\sum_{i=1}^{k}x_i^{r}P_i\right]^{1/r}\right)^{s}P|\mathbb{H}\right)^{1/s}\\
&= \left[Pf\left(\sum_{i=1}^{k}x_iP_i\right)^{s}P|\mathbb{H}\right]^{1/s}\\
&= \left[P\left(\sum_{i=1}^{k}f(x_i)P_i\right)^{s}P|\mathbb{H}\right]^{1/s}\\
&= \left(\sum_{i=1}^{k}Pf(x_i)^{s}P_iP|\mathbb{H}\right)^{1/s}\\
&= \left(\sum_{i=1}^{k}f(x_i)^{s}PP_iP|\mathbb{H}\right)^{1/s}\\
&= \left(\sum_{i=1}^{k}\alpha_if(x_i)^{s}A_i\right)^{1/s}.
\end{aligned}$$

Chapter 3

Matrix inequalities and the in-sphere property

The classical Cauchy inequality for m non-negative numbers $a_1, \cdots, a_m$ is stated as follows:

$$\frac{a_1 + a_2 + \cdots + a_m}{m} \geq \sqrt[m]{a_1 a_2 \cdots a_m}.$$

There are many reverse versions of the above inequality. One of the well-known reverse Cauchy inequalities is the following one:

$$\frac{a_1 + a_2 + \cdots + a_m}{m} \leq \sqrt[m]{a_1 a_2 \cdots a_m} + \frac{1}{m} \sum_{1 \leq i,j \leq m} |a_i - a_j|. \tag{3.0.1}$$

Besides, for $a, b \geq 0$ and $0 \leq s \leq 1$, the Young inequality is as follows:

$$sa + (1-s)b \geq a^s b^{1-s} \geq \frac{a + b - |a-b|}{2}.$$

At the same time, it is obvious that,

$$\min\{a, b\} = \frac{a+b}{2} - \frac{|a-b|}{2} \leq a^{1-s} b^s = a \sharp_s b, \quad \text{or} \quad \frac{a+b}{2} - a \sharp_s b \leq \frac{|a-b|}{2}, \tag{3.0.2}$$

where $a\sharp_s b := a^{1/2}(a^{-1/2}ba^{-1/2})^s a^{1/2}$ is the s-power mean of a and b. The following inequality for the Heinz mean is an immediate consequence of (3.0.2)

$$\frac{a+b}{2} - \frac{1}{2}(a^s b^{1-s} + a^{1-s}b^s) \leq \frac{|a-b|}{2}. \tag{3.0.3}$$

Recall that the arithmetic-geometric means inequality has a refinement given by

$$\sqrt{ab} \leq \frac{a^s b^{1-s} + a^{1-s}b^s}{2} \leq \frac{a+b}{2} \tag{3.0.4}$$

for all $s \in [0, 1]$.

Inequalities (3.0.2) and (3.0.3) geometrically mean that the curve $a\sharp_s b$ and $\dfrac{a^s b^{1-s} + a^{1-s}b^s}{2}$ $(s \in [0, 1])$ are contained inside the circle with center at $\dfrac{a+b}{2}$ and radius being equal to half of the distance between a and b. In other words, the power mean and the Heinz mean satisfy *the in-sphere property* with respect to the Euclidean distance.

In this chapter, we investigate matrix versions of (3.0.2) and (3.0.3). More precisely, in the first section of the chapter we consider generalized reverse Cauchy inequalities for two positive definite matrices A and B and show that generalized reverse Cauchy inequalities hold under the condition $AB + BA \geq 0$. Moreover, we also show that the generalized reverse Cauchy inequality and the generalized Powers-Størmer inequality holds with respect to the unitarily invariant norms under the same condition. In the second section we prove some reverse inequalities of the matrix Heinz means with respect to unitarily invariant norms. And the last section dedicates the in-sphere property for matrix means.

3.1 Generalized reverse arithmetic-geometric mean inequalities

The results of this section are taken from [50].

Young inequalities for two positive matrices are important in estimating some quantum quantities, such as the quantum Chernoff bound [59] and the Tsallis relative entropy [33]. More precisely, the following trace inequality for the exponential positive semi-definite matrices (also called the generalized Powers-Størmer inequality) was studied by Audenaert et al. [10]: for $0 \le \nu \le 1$,

$$\mathrm{Tr}(A + B - |A - B|) \le 2\,\mathrm{Tr}(A^{\nu} B^{1-\nu}).$$

Noting that $f(t) = t^{\nu} (0 \le \nu \le 1)$ is operator monotone, Trung Hoa Dinh, Minh Toan Ho and Hiroyuki Osaka formulated a more general inequality and showed in [53] that

$$\mathrm{Tr}(A + B - |A - B|) \le 2\,\mathrm{Tr}(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}})$$

holds for any operator monotone function f on $[0, \infty)$ with $f((0, \infty))$ and $g(t) = \frac{t}{f(t)}$ $(t \in (0, \infty))$, $g(0) = 0$.

Furuichi [32], however, showed that the trace inequality

$$\frac{1}{2}\,\mathrm{Tr}(A + B - |A - B|) \le \mathrm{Tr}(A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\frac{1}{2}} A^{\frac{1}{2}}) \tag{3.1.5}$$

is not true in general.

When $m = 2$ and $\nu = \dfrac{1}{2}$, a natural matrix form of the reverse arithmetic-geometric mean inequality (3.0.1) for two positive definite matrices A and B could be written as

$$\frac{A + B}{2} \le A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\frac{1}{2}} A^{\frac{1}{2}} + \frac{|A - B|}{2},$$

where $A \sharp B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\frac{1}{2}} A^{\frac{1}{2}}$ is the geometric mean of A, B.

In general, the last inequality has the following form

$$\frac{A+B}{2} - A\sigma_f B \leq \frac{|A-B|}{2}, \tag{3.1.6}$$

where $A\sigma_f B = A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$ is the operator mean corresponding to the function f in the sense of Kubo and Ando [61]. We call inequality (3.1.6) *the generalized reverse arithmetic-geometric mean (AGM) inequality*. Such inequalities were studied by many authors. For example, Fujii-Nakamura-Pečarić-Seo [31] showed that for a symmetric operator mean with the representing function f and matrices $0 < kI \leq A, B \leq MI$ (where $k < M$),

$$A\nabla B - A\sigma_f B \leq M\left(\frac{k\nabla M}{k\sigma M} - 1\right) I.$$

The main results in this section are as follows.

Theorem 3.1.1. *Let f be a strictly positive operator monotone function on $[0,\infty)$ with $f((0,\infty)) \subset (0,\infty)$ and $f(1) = 1$. Then for any positive semi-definite matrices A and B with $AB+BA \geq 0$,*

$$A + B - |A-B| \leq 2A\sigma_f B. \tag{3.1.7}$$

Proof. Let $A \in \mathbb{H}_n^+$. Since f is continuous, we may assume that A is invertible. Let $P = (A-B)_+$ and $Q = (A-B)_-$ are the positive and negative parts of $A-B$, respectively. Then

$$A - B = (A-B)_+ - (A-B)_- = P - Q,$$

and

$$|A-B| = (A-B)_+ + (A-B)_- = P + Q.$$

From the assumption,

$$A + B - (P+Q) = (A+B)^2 - |A-B|^2 = 2(AB+BA) \geq 0.$$

Consequently,

$$(A+B)^2 \geq |A-B|^2.$$

Since the function $t^{1/2}$ is operator monotone, therefore

$$[(A+B)^2]^{1/2} \geq (|A-B|^2)^{1/2} = |A-B|.$$

On other hand, $A-B = P-Q$ and $|A-B| = P+Q$, then

$$A-P = B-Q = \frac{A-P+B-Q}{2} = \frac{A+B-|A-B|}{2} \geq 0.$$

Moreover,

$$A-P = B-Q \leq B \text{ (since } Q \geq 0)$$

Hence,

$$A^{-\frac{1}{2}}(A-P)A^{-\frac{1}{2}} \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}}.$$

Consequently, by the operator monotonicity of the function f and the last inequality we have

$$f(A^{-\frac{1}{2}}(A-P)A^{-\frac{1}{2}}) \leq f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}).$$

By the monotonicity of σ_f, $A-P \leq A$ and $A-P \leq A-P$,

$$(A-P)\sigma_f(A-P) \leq A\sigma_f(A-P).$$

Then

$$\begin{aligned}
\frac{1}{2}(A+B-|A-B|) &= A-P \\
&= (A-P)\sigma_f(A-P) \\
&\leq A\sigma_f(A-P) \\
&\leq A\sigma_f B.
\end{aligned}$$

$\square$

The following example shows that the condition $AB + BA \geq 0$ is necessary.

Example 3.1.1. Let $f(t) = t^{\frac{1}{2}}$. Then σ_f is the geometric mean. For the following matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix},$$

with the help of Matlab, we have

$$2A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}} - (A + B - |A - B|) = \begin{pmatrix} 1.1463 & 0.7174 \\ 0.7174 & -0.0253 \end{pmatrix}.$$

This shows that the matrix inequality (3.1.7) is not generally true. Notice that in this case $\det(AB + BA) = -2 < 0$, which means that $AB + BA$ is not positive semi-definite.

Recall that an operator monotone function f is called *symmetric* if $f(t) = tf(t^{-1})$. It is well known [61] that a symmetric positive operator monotone function f with $f(1) = 1$ satisfies

$$\frac{2t}{1+t} \leq f(t) \leq \frac{1+t}{2}.$$

Then, for $A, B \geq 0$ and for a symmetric operator monotone function f,

$$f(A^{-1/2}BA^{-1/2}) \leq \frac{I + A^{-1/2}BA^{-1/2}}{2}.$$

Consequently,

$$A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2} \leq A^{1/2}\frac{I + A^{-1/2}BA^{-1/2}}{2}A^{1/2},$$

or

$$A\sigma_f B \leq \frac{A+B}{2}.$$

Even if f is not symmetric, we have the following proposition, which was kindly pointed out to us by Jun Ichi Fujii.

Proposition 3.1.1. *Let f be a strictly positive operator monotone function on $(0,\infty)$ with $f((0,\infty)) \subset (0,\infty)$ and $f(1)=1$. Then for any positive semi-definite matrices A and B*

$$2A\sigma_f B \leq A+B+|A-B|.$$

Proof. We use the arguments similar to those in the proof of Theorem 3.1.1.

Let $A-B=P-Q$, where $P=(A-B)_+$ and $Q=(A-B)_-$ are the positive and negative parts of $A-B$, respectively.

By the monotonicity of σ_f, $A \leq A+Q$ and $A-P+Q \leq A+Q$ then $A\sigma_f(A-P+Q) \leq (A+Q)\sigma_f(A+Q)$. Therefore,

$$\begin{aligned}
A\sigma_f B &= A\sigma_f(A-P+Q) \\
&\leq (A+Q)\sigma_f(A-P+Q) \\
&\leq (A+Q)\sigma_f(A+Q) \\
&= (A+Q)^{\frac{1}{2}} f((A+Q)^{-\frac{1}{2}}(A+Q)(A+Q)^{-\frac{1}{2}})(A+Q)^{\frac{1}{2}} \\
&= (A+Q)^{\frac{1}{2}} f(I)(A+Q)^{\frac{1}{2}} \\
&= A+Q \\
&= \frac{A+B+P+Q}{2} \\
&= \frac{A+B}{2} + \frac{|A-B|}{2}.
\end{aligned}$$

□

3.2 Reverse inequalities for the matrix Heinz means

The results of this section are taken from [52].

Recall that the reverse arithmetic-geometric mean inequalities (3.0.2) and (3.0.3) have matrix versions for positive definite matrices with any unitarily invariant norm $|||\cdot|||$ as follows [16]:

$$\left|\left|\left|A^{1/2}B^{1/2}\right|\right|\right| \leq \left|\left|\left|\frac{A^sB^{1-s}+A^{1-s}B^s}{2}\right|\right|\right| \leq \left|\left|\left|\frac{A+B}{2}\right|\right|\right|.$$

From (3.1.7), the following reverse inequality for the matrix Heinz mean holds: for any $A, B \in \mathbb{H}_n^+$ such that $AB + BA \geq 0$ and $s \in [0, 1]$,

$$\frac{A+B}{2} - \frac{1}{2}|A-B| \leq \frac{A\sharp_s B + A\sharp_{1-s}B}{2}. \tag{3.2.8}$$

For positive definite matrices A and B, another matrix version of Heinz mean is defined as $\dfrac{A\sharp_s B + A\sharp_{1-s}B}{2}$. However, without the condition $AB+BA \geq 0$, it was shown in [47, Theorem 2.1] that for any operator mean σ and for any $A, B \in \mathbb{P}_n$,

$$\frac{A+B}{2} - A\sigma B \leq \frac{1}{2}A^{1/2}\left|I - A^{-1/2}BA^{-1/2}\right|A^{1/2}. \tag{3.2.9}$$

The aim of the next subsections is to present new general reverse inequalities with (3.2.9) for unitarily invariant norms. As a consequence, we obtain a new reverse inequality for Heinz means.

3.2.1 Reverse arithmetic-Heinz-geometric mean inequalities with unitarily invariant norms

The results of this subsection are taken from [52].

Recall that a norm $|||\cdot|||$ on $\mathbb{M}_n$ is unitarily invariant if $|||UAV||| = |||A|||$ for any unitary matrices U, V and any $A \in \mathbb{M}_n$. Ky Fan Dominance Theorem [16] asserts that given $A, B \in \mathbb{M}_n$, $s(A) \prec_w s(B)$ if and only if $|||A||| \leq |||B|||$ for all unitarily invariant norms $|||\cdot|||$, where $s(A)$

denotes the vector of singular values of A.

An immediate consequence [47, Theorem 2.1] is as follows.

Lemma 3.2.1. *([47, Theorem 2.1]) Let σ and τ be arbitrary operator means and $|||\cdot|||$ be any unitarily invariant norm on $\mathbb{M}_n$. Then for any $A, B \in \mathbb{H}_n^+$,*

$$\left|\left|\left|\frac{A+B}{2} - \frac{1}{2}A^{1/2}|I - A^{-1/2}BA^{-1/2}|A^{1/2}\right|\right|\right| \leq |||A\sigma B|||$$

and

$$\left|\left|\left|\frac{A+B}{2} - \frac{1}{2}A^{1/2}|I - A^{-1/2}BA^{-1/2}|A^{1/2}\right|\right|\right| \leq \frac{1}{2}|||A\sigma B + A\tau B|||.$$

When σ is the Heinz mean, for any $s \in [0,1]$ we have

$$\left|\left|\left|\frac{A+B}{2} - \frac{1}{2}A^{1/2}|I - A^{-1/2}BA^{-1/2}|A^{1/2}\right|\right|\right| \leq \left|\left|\left|\frac{A\sharp_s B + A\sharp_{1-s}B}{2}\right|\right|\right|. \tag{3.2.10}$$

If $s > 1$, (3.2.10) is reversed.

It is also natural to consider the following matrix inequality

$$A\nabla B \leq A^{s/2}B^{1-s}A^{s/2} + \frac{1}{2}A^{1/2}|I - A^{-1/2}BA^{-1/2}|A^{1/2}, \tag{3.2.11}$$

where $s \in [0,1]$. The following example shows that the last matrix inequality does not hold for $s = 1/2$. Indeed, for the following positive definite matrices

$$A = \begin{pmatrix} 0.699 & 1.1455 \\ 1.1455 & 4.9308 \end{pmatrix}, \quad B = \begin{pmatrix} 0.9249 & 0.7064 \\ 0.7064 & 0.5928 \end{pmatrix},$$

the matrix

$$A^{1/4}B^{1/2}A^{1/4} + \frac{1}{2}A^{1/2}|I - A^{-1/2}BA^{-1/2}|A^{1/2} - \frac{A+B}{2}$$

has eigenvalues 1.2956 and -0.0234. Therefore, the inequality (3.2.11) is not true. However, the eigenvalues of $A^{1/4}B^{1/2}A^{1/4}$ are 0.1531 and 2.1184, and the eigenvalues of $\dfrac{A+B}{2} - \dfrac{1}{2}A^{1/2}|I -$

$A^{-1/2}BA^{-1/2}|A^{1/2}$ are 0.9665 and 0.0327. That means,

$$\frac{A+B}{2} - \frac{1}{2}A^{1/2}|I - A^{-1/2}BA^{-1/2}|A^{1/2} \prec_w A^{1/4}B^{1/2}A^{1/4}$$

or

$$\left|\left|\left|\frac{A+B}{2} - \frac{1}{2}A^{1/2}|I - A^{-1/2}BA^{-1/2}|A^{1/2}\right|\right|\right| \le \left|\left|\left|A^{1/4}B^{1/2}A^{1/4}\right|\right|\right|. \tag{3.2.12}$$

At the same time, from Lemma 3.2.1 one also can ask whether the following inequality is true

$$\left|\left|\left|\frac{A+B}{2} - \frac{1}{2}A^{1/2}|I - A^{-1/2}BA^{-1/2}|A^{1/2}\right|\right|\right| \le \left|\left|\left|\frac{A^sB^{1-s} + B^sA^{1-s}}{2}\right|\right|\right|. \tag{3.2.13}$$

In the following theorem we prove more general inequalities of (3.2.12) for operator monotone functions. As a consequence, we will give a proof of (3.2.13).

Theorem 3.2.1. *Let $|||\cdot|||$ be an arbitrary unitarily invariant norm on $\mathbb{M}_n$. Let f be an operator monotone function on $[0,\infty)$ with $f((0,\infty)) \subset (0,\infty)$ and $f(0) = 0$, and g a function on $[0,\infty)$ such that $g(t) = \frac{t}{f(t)}$ $(t \in (0,\infty))$ and $g(0) = 0$. Then for any $A, B \in \mathbb{P}_n$,*

$$\left|\left|\left|\frac{A+B}{2} - \frac{1}{2}A^{1/2}|I - A^{-1/2}BA^{-1/2}|A^{1/2}\right|\right|\right| \le \left|\left|\left|f(A)^{1/2}g(B)f(A)^{1/2}\right|\right|\right| \tag{3.2.14}$$

$$\le \left|\left|\left|f(A)g(B)\right|\right|\right|. \tag{3.2.15}$$

Proof. Let us prove the first inequality. Suppose that $A \le B$. Then,

$$A + B - A^{1/2}|I - A^{-1/2}BA^{-1/2}|A^{1/2} = 2A.$$

Since g is operator monotone, we have $f(A)^{-1/2}Af(A)^{-1/2} = g(A) \le g(B)$. Hence,

$$f(A)^{1/2}g(A)f(A)^{1/2} \le f(A)^{1/2}g(B)f(A)^{1/2},$$

or

$$A \leq f(A)^{1/2} g(B) f(A)^{1/2}.$$

Therefore,

$$|||A||| \leq \left|\left|\left| f(A)^{1/2} g(B) f(A)^{1/2} \right|\right|\right|.$$

Next, we consider the general case. For the operator $I - A^{-1/2}BA^{-1/2}$, let $P = (I - A^{-1/2}BA^{-1/2})_+$ and $Q = (I - A^{-1/2}BA^{-1/2})_-$.

Then,

$$I - A^{-1/2}BA^{-1/2} = P - Q$$

and

$$|I - A^{-1/2}BA^{-1/2}| = P + Q.$$

Consequently,

$$A - B = A^{1/2}PA^{1/2} - A^{1/2}QA^{1/2}$$

and

$$A^{1/2}|I - A^{-1/2}BA^{-1/2}|A^{1/2} = A^{1/2}PA^{1/2} + A^{1/2}QA^{1/2}.$$

It is obvious that $A - A^{1/2}PA^{1/2} \in \mathbb{H}_n^+$. Since $A - A^{1/2}PA^{1/2} = B - A^{1/2}QA^{1/2} \leq B$, we get

$$A - A^{1/2}PA^{1/2} \leq f(A - A^{1/2}PA^{1/2})^{1/2} g(B) f(A - A^{1/2}PA^{1/2})^{1/2}.$$

Thus,

$$\left|\left|\left| A - A^{1/2}PA^{1/2} \right|\right|\right| \leq \left|\left|\left| f(A - A^{1/2}PA^{1/2})^{1/2} g(B) f(A - A^{1/2}PA^{1/2})^{1/2} \right|\right|\right|.$$

Notice that

$$
\begin{aligned}
&\left|\left|\left|f(A-A^{1/2}PA^{1/2})^{1/2}g(B)f(A-A^{1/2}PA^{1/2})^{1/2}\right|\right|\right| \\
&\qquad = \left|\left|\left|f(A-A^{1/2}PA^{1/2})^{1/2}g(B)^{1/2}g(B)^{1/2}f(A-A^{1/2}PA^{1/2})^{1/2}\right|\right|\right| \\
&\qquad \le \left|\left|\left|g(B)^{1/2}f(A-A^{1/2}PA^{1/2})g(B)^{1/2}\right|\right|\right| \\
&\qquad \le \left|\left|\left|g(B)^{1/2}f(A)g(B)^{1/2}\right|\right|\right| \\
&\qquad \le \left|\left|\left|f(A)^{1/2}g(B)f(A)^{1/2}\right|\right|\right|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left|\left|\left|A+B-A^{1/2}|I-A^{-1/2}BA^{-1/2}|A^{1/2}\right|\right|\right| &= 2\left|\left|\left|A-A^{1/2}PA^{1/2}\right|\right|\right| \\
&\le 2\left|\left|\left|f(A)^{1/2}g(B)f(A)^{1/2}\right|\right|\right|.
\end{aligned}
$$

The second inequality immediately follows from the Hiai-Ando log-majorization theorem which states that $\left|\left|\left|A^{1/2}BA^{1/2}\right|\right|\right| \le |||AB|||$. □

Corollary 3.2.1. *Let $A, B \in \mathbb{P}_n$ and $s \in [0,1]$. Then,*

$$
\left|\left|\left|\frac{A+B}{2}-\frac{1}{2}A^{1/2}|I-A^{-1/2}BA^{-1/2}|A^{1/2}\right|\right|\right| \le \left|\left|\left|A^{1/2}B^{1/2}\right|\right|\right|.
$$

Now let us give another version of reverse inequality for the Heinz mean.

Corollary 3.2.2. *Let $A, B \in \mathbb{H}_n^+$ and $s \in [0,1]$. Then we have*

$$
\left|\left|\left|\frac{A+B}{2}-\frac{1}{2}A^{1/2}|I-A^{-1/2}BA^{-1/2}|A^{1/2}\right|\right|\right| \le \frac{1}{2}\left|\left|\left|A^sB^{1-s}+A^{1-s}B^s\right|\right|\right|. \tag{3.2.16}
$$

Proof. Since A, B are positive definite matrices, the function $f(s) = |||A^sB^{1-s}+A^{1-s}B^s|||$ is continuous convex on $[0,1]$, and twice differentiable on $(0,1)$ and $f'(1/2) = 0$ (see [16, p.

265]). Hence, $f(s)$ attains a minimum on $[0,1]$ at $s = 1/2$. That means,

$$|||A^s B^{1-s} + A^{1-s} B^s||| \geq 2|||A^{1/2} B^{1/2}|||, \quad s \in [0,1].$$

On account of Corollary 3.2.1, we get the desired inequality (3.2.16).

$\square$

Corollary 3.2.3. *For any $A, B \in \mathbb{H}_n^+$ such that $AB + BA \geq 0$ and $s \in [0,1]$, we get the following inequalities*

$$|||A + B - |A - B|||| \leq 2|||A^{1/2} B^{1/2}||| \tag{3.2.17}$$

and

$$|||A + B - |A - B|||| \leq 2|||A^s B^{1-s} + A^{1-s} B^s|||. \tag{3.2.18}$$

Proof. Inequality (3.2.17) follows from Theorem 3.1.1. We can prove (3.2.18) by using similar arguments as in the proof of Corollary 3.2.2. $\square$

Remark 3.2.1. For any $A, B \in \mathbb{P}_n$ and $s \in [0,1]$ by the Araki-Lieb-Thirring inequality

$$\mathrm{Tr}(A^{1/2}|I - A^{-1/2} B A^{-1/2}| A^{1/2}) \geq \mathrm{Tr}(|A - B|).$$

It follows from the Powers-Størmer inequality that

$$\mathrm{Tr}\left(\frac{A+B}{2}\right) - \frac{1}{2}\mathrm{Tr}(A^{1/2}|I - A^{-1/2} B A^{-1/2}| A^{1/2}) \leq \mathrm{Tr}(A^{s/2} B^{1-s} A^{s/2}) = \mathrm{Tr}(A^s B^{1-s}).$$

3.2.2 Reverse inequalities for the matrix Heinz mean with Hilbert-Schmidt norm

The results of this subsection are taken from [52]

It is obvious that for any positive numbers a and b,

$$(a+b)^2 - |a^2 - b^2| \leq (a^s b^{1-s} + a^{1-s} b^s)^2. \tag{3.2.19}$$

In this section, based on (3.2.19) and (3.0.2) we obtain some reverse inequalities for matrix Heinz mean with Hilbert-Schmidt norms.

Theorem 3.2.2. *For any $A, B \in \mathbb{H}_n^+$ and $X \in \mathbb{M}_n$, then*

$$||AX + XB||_2^2 - ||AX - XB||_2^2 \leq ||A^s XB^{1-s} + A^{1-s} XB^s||_2^2. \tag{3.2.20}$$

Proof. Since positive semi-definite matrices are unitarily diagonalizable, hence there are unitary matrices U, V such that $A = UDU^*$ and $B = VEV^*$, where

$$D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n), \quad E = \mathrm{diag}(\gamma_1, \cdots, \gamma_n).$$

If we put $Z = U^* XV = [z_{ij}]$, then

$$\begin{aligned} AX + XB &= U((\lambda_i + \gamma_j)z_{ij})V^*, \\ AX - XB &= U((\lambda_i - \gamma_j)z_{ij})V^*, \\ A^s XB^{1-s} + A^{1-s} XB^s &= U((\lambda_i^s \gamma_i^{1-s} + \lambda_i^{1-s}\gamma_i^s)z_{ij})V^*. \end{aligned}$$

It is obvious that

$$(a + b)^2 \leq (a - b)^2 + (a^s b^{1-s} + a^{1-s} b^s)^2.$$

Therefore,

$$\begin{aligned} ||AX + XB||_2^2 &= \sum_{i,j=1}^{n} (\lambda_i + \gamma_j)^2 |z_{ij}|^2 \\ &\leq \sum_{i,j=1}^{n} \left[(\lambda_i - \gamma_j)^2\right] + (\lambda_i^s \gamma_j^{1-s} + \lambda_i^{1-s}\gamma_j^s)^2)|z_{ij}|^2 \\ &= ||AX - XB||_2^2 + ||A^s XB^{1-s} + A^{1-s} XB^s||_2^2. \end{aligned}$$

□

3.3 The in-sphere property for operator means

The results of this section are taken from [52].

Let $0 \leq a \leq b$. Then for any x in $[a, b]$, we have

$$x - a \leq b - a.$$

We call this *in-betweenness property* . Notice that any mean of numbers has in-betweenness property. At the same time, we also have

$$\frac{a+b}{2} - x \leq \frac{b-a}{2}. \tag{3.3.21}$$

In other words, any x between a and b lies inside the circle with the center at the arithmetic mean $\dfrac{a+b}{2}$ and the radius equals to half of the distance between a and b. It is worth to mention that the Powers-Størmer inequality is one of matrix generalizations of (3.3.21).

In [9], Audenaert introduced a geometric alternative to monotonicity of weighted means the "in-betweenness" property. A matrix mean $M(A, B, t)$ ($t \in [0, 1]$) such that $M(A, B, 1) = A$ and $M(A, B, 0) = B$ is said to satisfy the "in-betweenness" property with respect to a metric δ on $\mathbb{H}_n^+$ if the assignment $t \mapsto \delta(A, M(A, B, t))$ defines a monotonically decreasing real function. He proved the monotone property for some families of non-Kubo-Ando means. Of particular importance for [9] are the weighted brethren of Bhagwat and Subramanian's power or binomial means [15],

$$M_p(A, B, t) = (tA^p + (1-t)B^p)^{1/p}, \qquad p \in \mathbb{R}.$$

It is worth noting that in the particular cases of $p = \pm 1$, $M_p(A, B, t)$ is a mean in the sense of Kubo-Ando. In spite of this, the power means with $p > 1$ have many important applications, e.g., in mathematical physics and in the theory of operator spaces, where they form the basis of certain generalizations of l_p norms to non-commutative vector-valued L_p spaces [21]. Audenaert also conjectured that the "in-betweenness" property may hold for $p > 2$.

In [49] Trung Hoa Dinh, Raluca Dumitru and Jose Franco provided an alternate proof of the fact that the weighted power means $M_p(A, B, t) = (tA^p + (1-t)B^p)^{1/p}(1 \leq p \leq 2)$ satisfy Audenaert's "in-betweenness" property for positive semi-definite matrices. They also show that the "in-betweenness" property holds with respect to any unitarily invariant norm for $p = 1/2$ and with respect to the Euclidean metric for $p = 1/4$. For Kubo-Ando means they show that the only Kubo-Ando symmetric mean satisfying the "in-betweenness" property with respect to any metric induced by a unitarily invariant norm is the arithmetic mean.

In this section we will study in-sphere property for operator means. Firstly, recall that from (3.2.9), for any operator mean σ and for any $A, B \in \mathbb{H}_n^+$ with $AB + BA \geq 0$,

$$\left|\left|\left|\frac{A+B}{2} - A\sigma B\right|\right|\right| \leq \frac{1}{2}|||A - B|||. \tag{3.3.22}$$

The last inequality means that whatever operator mean σ we take, the point $A\sigma B$ can not run out of the sphere with center at $\dfrac{A+B}{2}$ and the radius equals to $\dfrac{1}{2}|||A-B|||$. This is one of matrix versions of in-sphere property of matrix means. However, if we fix some operator mean σ which is different from the arithmetic mean, then we can find a couple of matrices A, B so that $A\sigma B$ runs away from the circle mentioned above.

In the next theorem, we provide a new characterization of the matrix arithmetic mean by the inequality (3.1.6).

Theorem 3.3.1. *Let σ be an arbitrary symmetric mean. If for any arbitrary unitarily invariant norm $|||\cdot|||$ on $\mathbb{M}_n$,*

$$\left|\left|\left|\frac{A+B}{2} - A\sigma B\right|\right|\right| \leq \frac{1}{2}|||A - B||| \tag{3.3.23}$$

whenever $A, B \in \mathbb{P}_n$, then σ is the arithmetic mean.

Proof. By [61, Theorem 4.4], the symmetric operator mean σ has the representation:

$$A\sigma B = \frac{\alpha}{2}(A+B) + \int_{(0,\infty)} \frac{\lambda+1}{\lambda}\{(\lambda A) : B + A : (\lambda B)\}d\mu(\lambda), \quad A, B \in \mathbb{P}_n. \tag{3.3.24}$$

where $\lambda \geq 0$ and μ is a positive measure on $(0, \infty)$ with $\alpha + \mu((0,\infty)) = 1$. For two orthogonal projections P, Q acting on a Hilbert space $\mathbb{H}$, $P \wedge Q$ denotes the orthogonal projection on the subspace $P(H) \cap Q(H)$. By [61, Theorem 3.7], we have

$$(\lambda P) : Q = P : (\lambda Q) = \frac{\lambda}{\lambda + 1} P \wedge Q.$$

Consequently, if $P \wedge Q$, then from (3.3.24) we get

$$P \sigma Q = \frac{\alpha}{2}(P + Q). \tag{3.3.25}$$

For $\theta > 0$ let us consider the following orthogonal projections

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}.$$

It is easy to see that $P \wedge Q = 0$. And then, for these projections on account of (3.3.25), the inequality (3.3.23) becomes

$$(1 - \alpha)|||P + Q||| \leq |||P - Q|||,$$

or

$$(1 - \alpha)|||P + Q||| \leq \sin\theta |||H|||, \tag{3.3.26}$$

where $H = \begin{pmatrix} \sin\theta & -\cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix}$. Since it is true for all $\theta > 0$, letting α tend to 0^+ from (3.3.26) we obtain $1 - \alpha \leq 0$. Thus, $\alpha \geq 1$. This shows that $\mu = 0$ and σ is the arithmetic mean. □

In the rest of the thesis we are going to show that if we replace the Kubo-Ando means by the power mean $M_p(A, B, t) = (tA^p + (1-t)B^p)^{1/p}$ with $p \in [1, 2]$ then the inequality in Theorem 3.3.1 holds without the condition $AB + BA \geq 0$. In other words, the matrix power means $M_p(A, B, t)$ satisfies in-sphere property with respect to the Hilbert-Schmidt 2-norm.

Theorem 3.3.2. *Let $p \in [1,2], t \in [0,1]$ and $M_p(A,B,t) = (tA^p + (1-t)B^p)^{1/p}$. Then for any pair of positive semi-definite matrices A and B,*

$$\left|\left|\frac{A+B}{2} - M_p(A,B,t)\right|\right|_2 \leq \frac{1}{2}\left|\left|A-B\right|\right|_2. \tag{3.3.27}$$

Proof. The following proof is based on the fact that for $p \in [1,2]$ the function $x^{1/p}$ is operator concave and the function $x^{2/p}$ is operator convex.

Since $||A||_2 = (\mathrm{Tr}(A^2))^{1/2}$, the inequality (3.3.27) is equivalent to the following

$$\begin{aligned}
\left[\mathrm{Tr}\left([\frac{A+B}{2} - M_p(A,B,t)]^2\right)\right]^{1/2} &\leq \frac{1}{2}\left[\mathrm{Tr}([A-B]^2)\right]^{1/2} \\
\mathrm{Tr}\left([\frac{A+B}{2} - M_p(A,B,t)]^2\right) &\leq \frac{1}{4}\,\mathrm{Tr}([A-B]^2) \\
\mathrm{Tr}\left(\frac{(A+B)^2}{4}\right) - \mathrm{Tr}\left[(A+B)M_p(A,B,t)\right] + \mathrm{Tr}\left(M_p^2(A,B,t)\right) &\leq \frac{1}{4}\left[\mathrm{Tr}([A+B]^2) - 4\,\mathrm{Tr}(AB)\right] \\
\mathrm{Tr}(M_p^2(A,B,t)) - \mathrm{Tr}((A+B)M_p(A,B,t)) &\leq -\,\mathrm{Tr}(AB).
\end{aligned} \tag{3.3.28}$$

It is obvious that inequality (3.3.28) holds for $t = 0$ or $t = 1$. We have to show that the set of t satisfying (3.3.28) is convex subset in $[0,1]$, and then it coincides with $[0,1]$. Let s,t belong to $[0,1]$ satisfying (3.3.28). We now show that (3.3.28) is true for $(t+s)/2$.

Note that

$$\begin{aligned}
M_p\left(A,B,\frac{t+s}{2}\right) &= \left(\frac{t+s}{2}A^p + (1-\frac{t+s}{2})B^p\right)^{1/p} \\
&= \left(\frac{1}{2}\left[tA^p + (1-t)B^p\right] + \frac{1}{2}[sA^p + (1-s)B^p]\right)^{1/p} \\
&= \left(\frac{1}{2}M_p^p(A,B,t) + \frac{1}{2}M_p^p(A,B,s)\right)^{1/p}.
\end{aligned}$$

Since $p \in (1,2)$ the function $x^{1/p}$ is operator concave, then we have

$$\begin{aligned} M_p\left(A,B,\frac{t+s}{2}\right) &= \left(\frac{1}{2}M_p^p(A,B,t)+\frac{1}{2}M_p^p(A,B,s)\right)^{1/p} \\ &\geq \frac{1}{2}M_p(A,B,t)+\frac{1}{2}M_p(A,B,s). \end{aligned}$$

Consequently, for the positive matrix $A+B$,

$$\operatorname{Tr}\left((A+B)M_p\left(A,B,\frac{t+s}{2}\right)\right) \geq \frac{1}{2}\operatorname{Tr}\left((A+B)M_p(A,B,t)+(A+B)M_p(A,B,s)\right). \tag{3.3.29}$$

In the other hand, for $p \in [1,2]$ the function $x^{2/p}$ is operator convex, then we have

$$\begin{aligned} M_p^2\left(A,B,\frac{t+s}{2}\right) &= \left[\frac{1}{2}M_p^p(A,B,t)+\frac{1}{2}M_p^p(A,B,s)\right]^{2/p} \\ &\leq \frac{1}{2}M_p^2(A,B,t)+\frac{1}{2}M_p^2(A,B,s). \end{aligned} \tag{3.3.30}$$

From (3.3.29) and (3.3.30) we obtain

$$\begin{aligned} &\operatorname{Tr}\left(M_p^2\left(A,B,\frac{t+s}{2}\right)\right)-\operatorname{Tr}\left((A+B)M_p\left(A,B,\frac{t+s}{2}\right)\right) \\ &\leq \frac{1}{2}\operatorname{Tr}\left(M_p^2(A,B,t)\right)+\frac{1}{2}\operatorname{Tr}(M_p^2(A,B,s))-\frac{1}{2}\operatorname{Tr}\left((A+B)M_p(A,B,t)\right) \\ &\quad -\frac{1}{2}\operatorname{Tr}\left((A+B)M_p(A,B,s)\right) \\ &\leq -\operatorname{Tr}(AB). \end{aligned}$$

Therefore, inequality (3.3.27) holds for $(s+t)/2$. $\square$

Conclusion

The thesis obtains the following results.

1. Define new class of operator (p, h)-convex functions and obtain properties for them. This is a new class of operator function, generalizing many classes of known operator functions.

2. Provide a type of Jensen inequality for operator (p, h)-convex function, generalizing for many types of Jensen inequality for known classes of operator convex functions

3. Provide a Hansen-Pedersen type inequality for operator(p, h)-convex functions, prove an inequality for index set functions for this class of function.

4. Define a class of operator (r, s)-convex function and study some properties for them. This is also a new class of operator convex functions, generalizing the class of operator r-convex functions.

5. Prove the Jensen and Rado type inequalities for operator (r, s)-convex functions.

6. Provide some equivalent conditions for a function to be operator (p, h)-convex and (r, s)-convex, respectively.

7. Prove a generalized reverse arithmetic-geometric mean inequality involving Kubo-Ando means.

8. Prove some reverse norm inequalities for the matrix Heinz mean.

9. Obtain a new characterization of the arithmetic mean by a matrix inequality with respect to the unitarily norm.

10. Obtain the ”in-sphere property” for matrix means with respect to unitary invariant norm and Hilbert Schmidt norm. At the same time, we also show that the matrix power mean satisfies the in-sphere property with respect to the Hilbert-Schmidt norm.

Future investigation.

In the near future, we intend to continue investigation in the following direction:

1. Continue to characterize new classes of operator convexity with some well-known matrix means.

2. Let p, q be positive numbers, h be super-multiplicative non-negative real valued function. A function f is called operator (p, h, q)-convex if

$$f\left([\alpha A^p + (1-\alpha)B^p]^{1/p}\right) \leq [h(\alpha)f(A)^q + h(1-\alpha)f(B)^q]^{1/q}$$

   If $q = 1$ then we get the class of operator $(p, h, 1)$-convex or called operator (p, h)-convex, and if $h \equiv id$ is identity function, we get the class of operator (p, id, q)-convex functions, or called as operator (r, s)-convex functions. In the future, we intend to continue to investigate this general class of operator functions for some different cases.

3. In-sphere property of the matrix mean: We believe that the matrix power mean satisfies in-sphere property with respect to the p-Schatten norm a larger range of p and for any unitarily invariant norm.

4. Define new classes of quantum entropy in relation with new types of operator convex functions. It would be meaningful to study their properties and applications in quantum information theory.

List of Author's Papers related to the thesis

1. D. T. Hoa, V. T. B. Khue, H. Osaka (2016), "A generalized reverse Cauchy inequality for matrices", *Linear and Multilinear Algebra.* 64, 1415-1423.

2. D. T. Hoa, V. T. B. Khue (2017), "Some inequalities for operator (p, h)-convex functions", *Linear and Multilinear Algebra.*

   http://dx.doi.org/10.1080/03081087.2017.1307914.

3. D. T. Hoa, D. T. Duc, V. T. B. Khue (2017), "A new type of operator convexity", accepted for publication in Acta Mathematica Vietnamica.

4. D. T. Hoa, V. T. B Khue, T.-Y. Tam (2017), "In-sphere property and reverse inequalities for the matrix Heinz mean", submitted.

Bibliography

[1] P. A. Absil, R. E. Mahony, R. Sepulchre (2007), *Optimization Algorithms on Matrix Manifolds*, Princeton.

[2] A. Aleman (1985), "On some generalizations of convex sets and convex functions", *Anal. Numer. Theor. Approx.*, 14(1), 1-6.

[3] W. N. Anderson, R. J. Duffin (1969), "Series and parallel addition of matrices", *J. Math. Anal. Appl.*, 26, 576-594.

[4] W. N. Anderson Jr. , G. E. Trapp (1975), "Shorted operators II", *SIAM J. Appl. Math*, 28, 60-71.

[5] T. Ando (1978), *Topics on operator inequalities*, Lecture Note, Sapporo.

[6] T. Ando (1979), "Concavity of certain maps on positive definite matrices and applications to Hadamard products", *Linear Algebra Appl.* , 26, 203-241.

[7] T. Ando, C.K. Li, R. Mathias (2004), "Geometric means", *Linear Algebra Appl.*, 385, 305-334.

[8] G. W. Anderson, M. K. Vamanamurthy, M. K. Vuorinen (2007), "Generalized convexity and inequalities", *J. Math. Anal. Appl.*, 335(2), 1294-1308.

[9] K. M. R. Audenaert (2013), "In-betweenness, a geometrical monotonicity property for operator means", *Linear Algebra Appl.*, 438(4), 1769-1778.

[10] K. M. R. Audenaert, J. Calsamiglia, Ll. Masanes, R. M Tapia, A. Acin, E. Bagan, F. Verstraete (2007), “Discriminating States: The Quantum Chernoff Bound”, *Phys. Rev. Lett.*, 98, 160501.

[11] T. Ando, F. Hiai (2011), “Operator log-convex functions and operator means”, *Math. Ann.*, 350(3), 611-630.

[12] K. M. R. Audenaert, F. Hiai (2013), “On matrix inequalities between the power means: Counterexamples”, *Linear Algebra Appl.*, 439, 1590-1604.

[13] M. Bakherad, H. Abbas, B. Mourad, M. S. Moslehian (2014), “Operator P-class functions”, *J. Inequal. Appl.*, 451.

[14] J. Bendat, S. Sherman (1955), “Monotone and convex operator functions”, *Trans. Amer. Math. Soc.*, 79, 58-71.

[15] K. V. Bhagwat, R. Subramanian (1978), “Inequalities between means of positive operators”, *Math. Proc. Camb. Phil. Soc.*, 83, 393-401.

[16] R. Bhatia (1997), *Matrix Analysis*, Springer, New York.

[17] R. Bhatia, J. Holbrook (2006), “Riemannian geometry and matrix geometric means”, *Linear Algebra Appl.*, 413(2-3), 594-618.

[18] D. A. Bini, B. Meini, F. Poloni (2010), “An effective matrix geometric mean satisfying the Ando-Li-Mathias properties”, *Math. Comput.*, 79, 437-452.

[19] W. W. Breckner (1978), “Stetigkeitsaussagen fureine Klasse verallgemeinerter knovexer funktionen in topologischen linearen Raumen”, *Publ. Inst. Math.*, 23, 13-20.

[20] E. A. Carlen (2010), “Trace inequalities and quantum entropy: An introductory course”, *Contemp. Math.*, 529, 73-140.

[21] E. A. Carlen, E. H. Lieb (1999), “A Minkowski type-trace inequality and strong subadditivity of quantum entropy”, *Amer. Math. Soc. Transl.*, 18(2), 59-69.

[22] F. Chen, X. Liu (2013), "Refinements on the Hermite Hadamard Inequalities for r-convex functions", *J. Appl. Math.*, 2013, 1-5.

[23] H. Chernoff (1952), "A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations", *Ann. Math. Stat.*, 4(23), 493-507.

[24] M.-D Choi (1974), "A Schwarz inequality for positive linear maps on C^*-algebras", *Illinois J. Math.*, 18, 565-574.

[25] C. Davis (1957), "A Schwarz inequality for convex operator functions", *Proc. Amer. Math. Soc.*, 8, 42-44.

[26] S. S. Dragomir (2001), "Refinements of the Hermite-Hadamard integral inequality for log-convex functions", *Aus. Math. Soc. Gazette*, 28(3), 129-133.

[27] S. S. Dragomir, C. E.M. Pearce (2000), *Selected Topics on Hermite Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University.

[28] S. S. Dragomir, J. Pečarić, L.-Erik Persson (1995), "Some inequalities of Hadamard type", *Soochow J. Math.*, 21, 335-341.

[29] Z. B. Fang, R. Shi (2014), "On the (p, h)-convex function and some integral inequalities", *J. Inequal. Appl.*, 45.

[30] J. I. Fujii, M. Kian, M. S. Moslehian (2010), "Operator Q-class functions", *Scientiae Mathematicae Japonicae*, 73(1), 75-80.

[31] J. I. Fujii, M. Nakamura, J. Pečarić, Y. Seo (2006), "Bounds for the ratio and difference between parallel sum and series via Mond-Pecaric method", *Math. Inequal. Appl.*, 9(4), 749-759.

[32] S. Furuichi (2011), "Inequalities for Tsallis relative entropy and generalized skew information", *Linear and Multilinear Alg.*, 59(10), 1143-1158.

[33] S. Furuichi (2012), “Refined Young inequalities with Specht’s ratio”, *J. Egyptian Math. Soc.*, 20(1), 46-49.

[34] S. Furuichi, K. Yanagi, K. Kuriyama (2004), “Fundamental properties of Tsallis relative entropy”, *J. Math. Phys.*, 45(12), 4868-4877.

[35] P.M. Gill, C.E.M. Pearce, Pečarić (1997), “Hadamard’s inequality for r-convex functions”, *J. Math. Anal. Appl.*, 215(2), 461-470.

[36] E. K. Godunova, V. I. Levin (1985), “Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions”, *Numerical Math. and Math. Phys.*, 166, 138-142.

[37] K. Guan (2010), “Multiplicative convexity and its applications”, *J. Math. Anal. Appl.*, 362, 156-166.

[38] K. Guan (2013), “GA-convexity and its applications”, *Analysis Math.*, 39, 189-208.

[39] F. Hansen (1980), “An operator inequality”, *Math. Ann.*, 246(3), 249-250.

[40] F. Hansen (2006), “Extensions of Lieb’s concavity Theorem”, *J. Stat. Physics*, 124(1), 87-101.

[41] F. Hansen, G. Ji, J. Tomiyama (2004), “Gaps between classes of matrix monotone functions”, *Bull. London Math. Soc.*, 36, 53-58.

[42] F. Hansen, G. K. Pedersen (1982), “An inequality for operators and Løwner theorem”, *Math. Ann.*, 258(3), 229-241.

[43] T. Hawkins (1975), “Cauchy and the spectral theory of matrices”, *Historia Math.*, 2(1), 1-29.

[44] E. Heinz (1951), “Beitrage zur Storungstheorie der Spektralzerlengung”, *Math. Ann.*, 123, 415-438.

[45] F. Hiai, M. Mosonyi, D. Petz, C. Beny (2011), "Quantum f-divergences and error correction", *Rev. Math. Phys.*, 23, 691-747.

[46] F. Hiai, D. Petz (2013), *Introduction to Matrix Analysis and Appl.*, Springer.

[47] D. T. Hoa (2015), "On characterization of operator monotonicity", *Linear Algebra Appl.*, 487, 260-267.

[48] D. T. Hoa, D. T. Duc, V. T. B. Khue (2017), "A new type of operator convexity ", accepted for publication in Acta Mathematica Vietnamica.

[49] D. T. Hoa, R. Dumitru, J. A. Franco (2017), "On the monotonicity of weighted power means for matrices", *Linear Algebra Appl.*, 527, 128-140.

[50] D. T. Hoa, V. T. B. Khue, H. Osaka (2016), "A generalized reverse Cauchy inequality for matrices", *Linear and Multilinear Alg.*, 64, 1415-1423.

[51] D. T. Hoa, V. T. B. Khue (2017), "Some inequalities for operator (p, h)-convex functions", *Linear and Multilinear Alg.* (http://dx.doi.org/10.1080/03081087.2017.1307914).

[52] D. T. Hoa, V. T. B Khue, T.-Y. Tam (2017), "In-sphere property and reverse inequalities for the matrix Heinz mean", submitted.

[53] D. T. Hoa, H. Osaka, H. M. Toan (2013), "On generalized Powers-Størmer's Inequality", *Linear Algebra Appl.*, 438(1), 242-249.

[54] D. T. Hoa, H. Osaka, J. Tomiyama (2015), "Characterization of operator monotone functions by Powers-Størmer type inequalities", *Linear and Multilinear Alg.*, 63(8), 1577-1589.

[55] R. A. Horn, C. R. Johnson (1985), *Matrix Analysis*, Cambridge University Press.

[56] J. Jensen (1906), "Sur les fonctions convexes et les inégalités entre les valeurs moyennes", *Acta Math.*, 30(1), 175-193.

[57] R. V. Kadison (1952), “A generalized Schwarz inequality and algebraic invariants for operator algebras”, *Ann. Math.* 56, 494-503.

[58] M. Kian, M. S. Moslehian (2015), “Operator Inequalities related to Q-Class functions”, *Math. Slovaca*, 65(1), 179-190.

[59] F. Kittaneh, Y. Manasrah (2010),“Improved Young and Heinz inequality for matrices”, *J. Math. Anal. Appl.*, 361, 262-269.

[60] F. Kraus (1936), “Überkonvexe Matrixfunktionen”, *Math. Z.* 41, 18-42.

[61] F. Kubo, T. Ando (1980), “Means of positive linear operators”, *Math. Ann.*, 246(3), 205-224.

[62] M. S. Moslehian, M. Kian (2012), “Jensen type inequalities for Q-class functions”, *Bull. Aust. Math. Soc.*, 85(1), 128-142.

[63] H. S. Lee, Y. D. Lim, T. Yamazaki (2011), “Multi-variable weighted geometric means of positive definite matrices”, *Linear Algebra Appl.*, 435, 307-322.

[64] E. H. Lieb (1973), “Convex Trace Functions and the Wigner-Yanase-Dyson Conjecture”, *Advances in Math.*, 11, 267-288.

[65] E. H. Lieb, M. B. Ruskai (1973), “Proof of the strong subadditivity of quantum-mechanical entropy”, *J. Math. Phys.*, 14, 1938-1941.

[66] C. Loewner (1934), “Über monotone Matrixfunctionen”, *Math. Z.*, 38, 177-216.

[67] M. Moakher (2005), “A differential-geometric approach to the geometric mean of symmetric positive-definite matrices”, *SIAM J. Matrix Anal. Appl.*, 26(3), 735-747.

[68] S. R. Mohan, S. K. Neogy (1995), “On invex sets and preinvex functions”, *J. Math. Anal. Appl.*, 189(3), 901-908.

[69] E. A. Morozova, N. N. Chentsov (1991), “Markov invariant geometry on state manifolds”, *J. Soviet Math.*, 56(5), 2648-2669.

[70] M. S. Moslehian, M. Kian (2012), “Jensen type inequalities for Q-class functions”, *Bull. Aust. Math. Soc.*, 85, 128-142.

[71] M. A. Naimark (1943), “On a representation of additive operator set functions”, *Dokl. Akad. Nauk SSSR*, 41(9), 373-375. (Russian); English translation: C.R. (Doklady) Akad. Sci. USSR. (N.S.) 41, 359-361.

[72] M. A. Nielsen, I. L. Chuang (2011), *Quantum Computation and Quantum Information*, Cambridge Univ. Press.

[73] M. A. Nielsen, D. Petz (2005), “A simple proof of the strong subadditivity inequality”, *Quantum Inf. Comput.*, 5(6), 507-513.

[74] Y. Ogata (2011), “A Generalization of Powers-Størmer Inequality”, *Let. Math. Phys.*, 97(3), 339-346.

[75] H. Osaka, S. Silvestrov, J. Tomiyama (2007), “Monotone operator functions, gaps and power moment problem”, *Math. Scand.*, 100(1), 161-183.

[76] CEM. Pearce, AM. Rubinov (1999), “p-functions, quasi-convex functions and Hadamard-type inequalities”, *J. Math. Anal. Appl.*, 240, 92-104.

[77] M. Pálfia (2016), “Operator means of probability measures and generalized Karcher equations”, *Advances in Math.*, 289, 951-1007.

[78] D. Petz (1996), “Monotone metrics on matrix spaces”, *Linear Algebra Appl.* 244(1), 81-96.

[79] W. Pusz, S. Woronowicz (1975), “Functional calculus for sesquilinear forms and the purification map”, *Rep. Math. Phys.*, 8, 159-170.

[80] M. B. Ruskai (2007), “Another short and elementary proof of strong subadditivity of quantum entropy”, *Rep. Math. Phys.*, 60(1), 1-12.

[81] O. E. Tikhonov (2006), “A note on definition of matrix convex functions”, *Linear Algebra Appl.*, 416(2-3), 773-775.

[82] KL. Tseng, GS. Yang, SS. Dragomir (2003), “ON quasi-convex functions and Hadamard’s inequality”, *RGMIA Res. Rep. Collect*, 6(3), Article ID 1.

[83] M. Uchiyama (1993), “Commutativity of selfadjoint operators”, *Pac. J. Math.*, 161, 385-392.

[84] S. Varošanec (2007), “On h-convexity”, *J. Math. Anal. Appl.*, 326(1), 303-311.

[85] X. Zhan (2002), *Matrix Inequalities*, Springer.

[86] G. Zabandan, A. Bodaghi, A. Kiliçmann (2012), “The Hermite-Hadamard inequality for r-convex functions”, *J. Ineq. Appl.*, 215.

[87] K. Zhang, J. Wan (2007), “p-convex functions and their properties”, *Pure Appl. Math.*, 23(1), 130-133.

[88] X. Zhang, G. Wang, Y. Chu (2009), “Convexity with respect to Hölder mean involving zero-balanced hypergeometric functions”, *J. Math. Anal. Appl.*, 353(1), 256-259.

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