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OPERATOR CONVEX FUNCTIONS, MATRIX INEQUALITIES AND SOME RELATED TOPICS

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Declaration

This thesis was completed at the Department of Mathematics, Quy Nhon University under the supervision of Assoc. Prof. Dr. Dinh Thanh Duc and Dr. Dinh Trung Hoa. I hereby declare that the results presented in it are new and original. Most of them were published in peer-reviewed journals, others have not been published elsewhere. For using results from joint papers I have gotten permissions from my co-authors.

Binh Dinh, 2018

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Contents

Glossary of Notation

$\mathbb{C}^n$	:	The linear space of all n-tuples of complex numbers
$\langle x, y \rangle$	:	The scalar product of vectors x and y
$\mathbb{M}_n$	:	The space of $n \times n$ complex matrices
$\mathbb{H}$	:	The Hilbert space
$\mathbb{H}_n$	:	The set of all $n \times n$ Hermitian matrices
$\mathbb{H}_n^+$	:	The set of $n \times n$ positive semi-definite matrices
$\mathbb{P}_n$	:	The set of positive definite matrices
I, O	:	The identity and zero elements of $\mathbb{M}_n$, respectively
A^*	:	The conjugate transpose (or adjoint) of the matrix A
$\lvert A \rvert$	:	The positive semi-definite matrix $(A^*A)^{1/2}$
$\mathrm{Tr}(A)$	:	The canonical trace of matrix A
$\lambda(A)$	:	The eigenvalue of matrix A
$\sigma(A)$	:	The spectrum of matrix A
$\lVert A \rVert$	:	The operator norm of matrix A
$\lvert\lvert\lvert A \rvert\rvert\rvert$	:	The unitarily invariant norm of matrix A
$x \prec y$	:	x is majorized by y
$A \sharp_t B$	:	The t-geometric mean of two matrices A and B
$A \sharp B$	:	The geometric mean of two matrices A and B
$A \nabla B$	:	The arithmetic mean of two matrices A and B
$A!B$	:	The harmonic mean of two matrices A and B
$A : B$	:	The parallel sum of two matrices A and B
$M_p(A, B, t)$	:	The matrix p-power mean of matrices A and B
$opgx(p, h, K)$	:	The class of operator (p, h)-convex functions on K
A_+, A_-	:	The positive and the negative parts of matrix A

Introduction

Nowadays, the importance of matrix theory has been well-acknowledged in many areas of engineering, probability and statistics, quantum information, numerical analysis, and biological and social sciences. In particular, positive definite matrices appear as data points in a diverse variety of settings: co-variance matrices in statistics [20], elements of the search space in convex and semi-definite programming [1] and density matrices in the quantum information [72].

In the past decades, matrix analysis becomes an independent discipline in mathematics due to a great number of its applications [5, 7, 18, 24, 25, 26, 27, 34, 39, 46, 85]. Topics of matrix analysis are discussed over algebras of matrices or algebras of linear operators in finite dimensional Hilbert spaces. Algebra of all linear operators in a finite dimensional Hilbert space is isomorphic to the algebra of all complex matrices of the same dimension. One of the main tools in matrix analysis is the spectral theorem in finite dimensional cases. Numerous results in matrix analysis can be transferred to linear operators on infinite dimensional Hilbert spaces without any difficulties. At the same time, many important results from matrices are not true so far for operators in infinite dimensional Hilbert spaces. Recently, many areas of matrix analysis are intensively studied such as theory of matrix monotone and matrix convex functions, theory of matrix means, majorization theory in quantum information theory, etc. Especially, physical and mathematical communities pay more attention on topics of matrix inequalities and matrix functions because of their useful applications in different fields of mathematics and physics as well. Those objects are also important tools in studying operator theory and operator algebra theory as well.

In 1930 von Neumann introduced a mathematical system of axioms of the quantum mechanics as follows:

(i) Each finite dimensional quantum system of n particles is associated with a Hilbert space of dimension 2^n;

(ii) Each observable in such a quantum system corresponds to a Hermitian matrix of the same dimension;

(iii) Each quantum state is associated to a density matrix, i.e., a positive semi-definite matrix of trace 1.

Therefore, matrix theory, matrix analysis and operator theory become the backgrounds of quantum mechanics and hence, several problems in quantum mechanics could be translated to others in the language of matrices. On the other hand, in the last decades along with an intensive development of the quantum information theory, matrix analysis becomes more popular and important.

Recall that if $\lambda_1, \lambda_2, \cdots, \lambda_k$ are eigenvalues of a Hermitian matrix A, then A can be represented as

$$A = \sum_{j=1}^{k} \lambda_j P_j,$$

where P_j is the orthogonal projection onto the subspace spanned by the eigen-vectors corresponding to the eigenvalue λ_j. And for a real-valued function f defined at λ_i ($i = 1, \cdots, k$), the matrix $f(A)$ is well-defined by *the spectral theorem* [43] as

$$f(A) = \sum_{j=1}^{k} f(\lambda_j) P_j. \tag{0.0.1}$$

In quantum theory most of important quantum quantities are defined with the canonical trace Tr on the algebra of matrices. An important quantity is the quantum entropy. For a density matrix A, *the quantum entropy* of A is the value

$$-\operatorname{Tr}(A\log(A)),$$

where the matrix $\log(A)$ is defined by (0.0.1).

It is worth to mention that the function $\log t$ is matrix monotone on $(0,\infty)$, while the function $t\log t$ is matrix convex on $(0,\infty)$. Recall that a function f is operator monotone on $(0,\infty)$ if and only if $tf(t)$ is operator convex on $(0,\infty)$. Operator monotone functions were first studied by K. Loewner in his seminal papers [66] in 1930. In the same decade, F. Krauss introduced operator convex functions [60]. Nowadays, the theory of such functions is intensively studied and becomes an important topic in matrix theory because of their vast of applications in matrix theory and quantum theory as well [41, 54, 55, 57, 63, 65, 69, 73, 75].

In general, a continuous function f defined on $K\subset\mathbb{R}$ is said to be [14]:

- *matrix monotone of order* n if for any Hermitian matrices A and B of order n with spectra in K,

$$A\le B \quad\Longrightarrow\quad f(A)\le f(B). \tag{0.0.2}$$

- *matrix convex of order* n if for any Hermitian matrices A and B of order n with spectra in K, and for any $0\le\lambda\le 1$,

$$f(\lambda A+(1-\lambda)B)\le\lambda f(A)+(1-\lambda)f(B). \tag{0.0.3}$$

If the function f is *matrix monotone* (*matrix convex*, respectively) for any dimension of matrices, then it is called *operator monotone* (*operator convex*, respectively).

An important example of operator monotone and convex functions is $f(t)=t^s$. Loewner showed that this function is operator monotone on $\mathbb{R}^+$ if and only if the power $s\in[0,1]$ while it is operator convex on $(0,\infty)$ if and only if $s\in[-1,0]\cup[1,2]$.

Now let us look back at the scalar mean theory which sets a starting point for our study in this thesis.

A *scalar mean* M of non-negative numbers is a function from $\mathbb{R}^+\times\mathbb{R}^+$ to $\mathbb{R}^+$ such that:

1) $M(x,x)=x$ for every $x\in\mathbb{R}^+$;
2) $M(x,y)=M(y,x)$ for every $x,y\in\mathbb{R}^+$;
3) If $x<y$, then $x<M(x,y)<y$;
4) If $x<x_0$ and $y<y_0$, then $M(x,y)<M(x_0,y_0)$;
5) $M(x,y)$ is continuous;
6) $M(tx,ty)=tM(x,y)$ for $t,x,y\in\mathbb{R}^+$.

A two-variable function $M(x,y)$ satisfying condition 6) can be reduced to a one-variable function $f(x):=M(1,x)$. Namely, $M(x,y)$ is recovered from f as $M(x,y)=xf(x^{-1}y)$. Notice that the function f, corresponding to M is monotone increasing on $\mathbb{R}^+$. And this relation forms a one-to-one correspondence between means and monotone increasing functions on $\mathbb{R}^+$.

In the last few decades, there has been a renewed interest in developing the theory of means for elements in the subset $\mathbb{H}_n^+$ of positive semi-definite matrices in the algebra $\mathbb{M}_n$ of all matrices of order n. Motivated by a study of electrical network connections, Anderson and Duffin [3] introduced a binary operator $A : B$, called parallel addition, for pairs of positive semi-definite matrices. Subsequently, Anderson and Trapp [4] have extended this notion to positive linear operators on a Hilbert space and demonstrated its importance in operator theory. Besides, the problem to find a *matrix analog of the geometric mean of non-negative numbers* was a long-standing problem since the product of two positive semi-definite matrices is not always a positive semi-definite matrix. In 1975, Pusz and Woronowicz [79] solved this problem and showed that the geometric mean $A\sharp B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ of two positive definite matrices A and B is the unique solution of the matrix Riccati equation

$$XA^{-1}X = B.$$

In 1980, Ando and Kubo [61] developed an axiomatic theory of operator means on $\mathbb{H}_n^+$. A binary operation σ on the class of positive operators, $(A, B) \mapsto A\sigma B$, is called a *connection* if the following requirements are fulfilled:

(i) Monotonicity: $A \leq C$ and $B \leq D$ imply $A\sigma B \leq C\sigma D$;

(ii) Transformation: $C^*(A\sigma B)C \leq (C^*AC)\sigma(C^*BC)$;

(iii) Continuity: $A_m \downarrow A$ and $B_m \downarrow B$ imply $A_m\sigma B_m \downarrow A\sigma B$ ($A_m \downarrow A$ means that the sequence A_m converges strongly in norm to A).

A *mean* σ is a connection satisfying the normalized condition:

(iv) $I\sigma I = I$ (where I is the identity element of $\mathbb{M}_n$).

The main result in Kubo-Ando theory is the proof of the existence of an affine order-isomorphism from the class of operator means onto the class of positive operator monotone functions on $\mathbb{R}^+$ which is described by

$$A\sigma_f B = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}.$$

This formula verifies that the geometric mean defined by Pusz and Woronowicz was natural and corresponding to the operator monotone function $f(t) = t^{1/2}$. A mean σ is called *symmetric* if $A\sigma B = B\sigma A$ for any positive matrices A and B. Or, equivalently, the representing function f of a symmetric mean satisfies $f(t) = tf(t^{-1})$, $t \in (0, \infty)$.

Later, motivated by information geometry, Morozova and Chentsov [69] studied monotone inner products under stochastic mappings on the space of matrices and monotone metrics in quantum theory. In 1996, Petz [78] proved that there is a correspondence between monotone metrics and operator means in the sense of Kubo and Ando, and hence, connected three important theories in quantum information theory and matrix analysis.

It is worth to mention that along with the quantum entropy of quantum states, many other important quantum quantities are defined with operator means, operator convex functions and the canonical trace.

Example 0.0.1. For two density matrices A and B, the *quantum relative entropy* [64] of A with respect to B is defined by

$$S(A||B) = -\operatorname{Tr}(A(\log A - \log B)).$$

The *quantum Chernoff bound* [10] in quantum hypothesis testing theory is given by a simple expression: For positive semi-definite matrices A and B,

$$Q(A, B) = \min_{0\leq s\leq 1}\{\operatorname{Tr}(A^sB^{1-s})\}.$$

One of important quantities in quantum theory is the *Renyi divergence* [20]: for $\alpha \in (0,1) \cup (1,\infty)$,

$$D_\alpha(A||B) = \frac{1}{\alpha - 1} \log \frac{\mathrm{Tr}(A^s B^{1-s})}{\mathrm{Tr}(A)}, \quad D_1 = \frac{\mathrm{Tr}(A(\log A - \log B))}{\mathrm{Tr}(A)}.$$

All of quantities listed above are special cases of the quantum f-divergence in quantum theory where f is some operator convex function [45]. Thus, the theory of matrix functions is an important part of matrix analysis and of quantum information theory as well.

Now let σ and τ be arbitrary operator means (not necessarily Kubo-Ando means) [61]. We introduce a general approach to operator convexity as follows.

A non-negative continuous function f on $\mathbb{R}^+$ is called *$\sigma\tau$-convex* if for any positive definite matrices A and B,

$$f(A\sigma B) \leq f(A)\tau f(B). \tag{0.0.4}$$

When σ and τ are the arithmetic mean, the function f satisfying the above inequality is operator convex. When σ is the arithmetic mean and τ is the geometric mean, the function f satisfying (0.0.4) is called *operator* $\log$*-convex*. Such functions were fully characterized by Hiai and Ando in [11] as decreasingly monotone operator functions.

The matrix power mean of positive semi-definite matrices A and B was first studied by Bhagwat and Subramanian [15] as

$$M_p(A,B,t) = (tA^p + (1-t)B^p)^{1/p}, \quad \text{for} \quad p \in \mathbb{R}.$$

The matrix power mean $M_p(A,B,t)$ is a Kubo-Ando mean if and only if $p = \pm 1$. Nevertheless, the power means with $p > 1$ have many important applications in mathematical physics and in the theory of operator spaces [21].

In this thesis, we use (0.0.4) to define some new classes of operator convex functions with the matrix power means $M_p(A,B,t)$. We study properties of such functions and prove some well-known inequalities for them. We also provide several equivalent conditions for a function to be operator convex in this new sense.

Now, let us consider some geometrical interpretations for scalar means and matrix means. Let $0 \leq a \leq x \leq b$. It is obvious that the arithmetic mean $(a+b)/2$ is the unique solution of the optimization problem

$$(x-a)^2 + (x-b)^2 \to \min, \quad x \in \mathbb{R}.$$

And for any scalar mean M on $\mathbb{R}^+$,

$$M(a,b) - a \leq b - a.$$

We call this *the in-betweenness property*.

In 2013, Audenaert studied the in-betweenness property for matrix means in [9]. Recently, Dinh, Dumitru and Franco [49] continued to investigate this property for the matrix power means. They provided some partial solutions to Audenaert's conjecture in [9] and a counterexample to the conjecture for $p > 0$.

From the property 3) in the definition of scalar means, it is obvious that,

$$\frac{a+b}{2} - M(a,b) \leq \frac{b-a}{2}. \tag{0.0.5}$$

In other words, $M(a,b)$ lies inside the sphere centered at the arithmetic mean $\dfrac{a+b}{2}$ with the radius equal to a half of the distance between a and b. We call this *the in-sphere property* of scalar means with respect to the Euclidean distance on $\mathbb{R}$. Notice that for $s \in [0,1]$ and $p > 0$ the s-weighted geometric mean $M(a,b) = a^{1-s}b^s$ and the power mean (or binomial mean) $M_p(a,b,s) = ((1-s)a^p + sb^p)^{1/p}$ satisfy the in-sphere property (0.0.5).

Now, let A and B be positive definite matrices. The Riemannian distance function on the set of positive definite matrices is defined by

$$\delta_R(A,B) = \left(\sum_i \log^2(\lambda_i(A^{-1}B))\right)^{1/2}.$$

In 2005, Moakher [67] showed that the geometric mean $A\sharp B$ is the unique minimizer of the sum of the squares of the distances:

$$\delta_R^2(X,A) + \delta_R^2(X,B) \to \min, \quad X \geq 0.$$

Almost at the same time, Bhatia and Holbrook [17] showed that the curve

$$\gamma(s) = A\sharp_s B := A^{1/2}(A^{-1/2}BA^{-1/2})^s A^{1/2} \quad (s \in [0,1])$$

is the unique geodesic (i.e., the shortest) path joining A and B. Furthermore, the geometric mean $A\sharp B$ is the midpoint of this geodesic. Therefore, the picture for matrix means is very different from the one for scalar ones.

Notice that one of the important matrix generalizations of the in-sphere property is the famous Powers-Størmer inequality proved by Audenaert et. al. [10], and then expanded to operator algebras by Ogata [74]: for any positive semi-definite matrices and for any $s \in [0,1]$,

$$\operatorname{Tr}(A+B-|A-B|) \leq 2\operatorname{Tr}(A^s B^{1-s}). \tag{0.0.6}$$

Using the last inequality the authors solved a problem in quantum hypothesis testing theory: to define the quantum generalization of the Chernoff bound [23]. The quantity on the left hand side of (0.0.6) is called *the non-logarithmic quantum Chernoff bound*. Along with the mentioned above importance of matrix means, the Powers-Størmer inequality again shows us that the picture of matrix means is really interesting and complicated.

The second aim of this thesis is to investigate various matrix versions of in-sphere property (0.0.5). More precisely, we study inequalities involving matrices, matrix means, trace, norms and matrix functions. We also consider the in-sphere property for matrix means with respect to some distance functions on the manifold of positive semi-definite matrices.

The purposes of the current thesis are as follows.

1. Investigate new types of operator convex functions with respect to matrix means, study their properties and prove some well-known inequalities for them.

2. Characterize new types of operator convex functions by matrix inequalities.

3. Study reverse arithmetic-geometric means inequalities involving general matrix means.

4. Study reverse inequalities for the matrix Heinz means and unitarily invariant norms.

5. Study in-sphere properties for matrix means with respect to unitarily invariant norms.

Methodology. The main tool in our research is the spectral theorem for Hermitian matrices. We use techniques in the theory of matrix means of Kubo and Ando to define new types of operator convexity. Some basic techniques in the theory of operator monotone functions and operator convex functions are also used in the dissertation. We also use basic knowledge in matrix theory involving unitarily invariant norms, trace, etc.

Main results of the work were presented on the seminars at the Department of Mathematics at Quy Nhon University and on international workshops and conferences as follows:

1. The Second Mathematical Conference of Central- Highland of Vietnam, Da Lat University, November 2017.
2. The 6th International Conference on Matrix Analysis and Applications, Duy Tan University, June 2017.
3. Conference on Algebra, Geometry and Topology, Dak Lak Pedagogical College, November 2016.
4. International Workshop on Quantum Information Theory and Related Topics, VIASM, September 2015.
5. Conference on Mathematics of Central-Highland Area of Vietnam, Quy Nhon University, August 2015.
6. Conference on Algebra, Geometry and Topology (DAHITO), Ha Long, December 2014.
7. International Workshop on Quantum Information Theory and Related Topics, Ritsumeikan University, Japan, September 2014.

This thesis has Introduction, three chapters, Conclusion, a list of the author's papers related to the thesis and preprints related to the topics of the thesis, and a list of references.

Brief content of the thesis.

In Introduction the author provides a background on the topics which are considered in this work. The meaningfulness and motivations of this work are explained. The author also provides a brief content of the thesis with main results from the last two chapters.

In the first chapter the author collects some basic preliminaries which are used in the thesis.

In the second chapter the author defines and studies new classes of operator convex functions, their properties, proves some well-known inequalities for them and obtains a series of characterizations.

In the third chapter, we study the in-sphere property for matrix means. We also establish some reverse inequalities for the matrix Heinz means and provide a new characterization of the matrix arithmetic mean.

Chapter 1

Preliminaries

Let $\mathbb{N}$ be the set of all natural numbers. For each $n \in \mathbb{N}$, we denote by $\mathbb{M}_n$ the algebra of all $n \times n$ complex matrices. Denote by I and O the identity and zero elements of $\mathbb{M}_n$, respectively. In this thesis we consider problems for matrices, i.e., operators in finite dimensional Hilbert spaces. We will mention if the case is infinite dimensional.

Recall that for two vectors $x = (x_j), y = (y_j) \in \mathbb{C}^n$ the *inner product* $\langle x, y\rangle$ of x and y is defined as $\langle x, y\rangle \equiv \sum_j x_j \overline{y}_j$. Now let A be a matrix in $\mathbb{M}_n$. The *conjugate transpose* or the *adjoint* A^* of A is the complex conjugate of the transpose A^T. We have, $\langle Ax, y\rangle = \langle x, A^*y\rangle$.

A matrix A is called:

– *self-adjoint* or *Hermitian* if $A = A^*$, or, it is equivalent to that $\langle Ax, y\rangle = \langle x, Ay\rangle$;
– *unitary* if $AA^* = A^*A = I$;
– *positive semi-definite* (or positive) (we write $A \geq 0$) if

$$\langle x, Ax\rangle \geq 0 \quad \text{for all} \quad x \in \mathbb{C}^n; \tag{1.0.1}$$

– *positive definite* (or strictly positive) (we write $A > 0$) if (1.0.1) is strict for all non-zero vector $x \in \mathbb{C}^n$;
– *orthogonal projection* if $A = A^* = A^2$.

Note that in the finite dimensional case, $A > 0$ if and only if A is invertible and $A \geq 0$. A positive semi-definite matrix is necessary Hermitian. Further, we denote by $\mathbb{H}_n$ the set of all $n \times n$ Hermitian matrices, by $\mathbb{H}_n^+$ and $\mathbb{P}_n$ the $n \times n$ positive semi-definite and positive definite matrices, respectively. Mention that for any matrix A, the matrix A^*A is always positive semi-definite. Hence, as a consequence of (v), *the module* $|A|$ of A is well defined by $|A| := (A^*A)^{1/2}$. The partial order (the *Loewner partial order*) on the set $\mathbb{H}_n$ of Hermitian matrices as follows:

$$A \geq B \quad \text{if} \quad A - B \geq 0.$$

A positive semi-definite matrix A with trace 1 is called *a density matrix* which is associated with a quantum state in some quantum system. In this sense, all rank one orthogonal projections in $\mathbb{M}_n$ are called *pure states*. And positive semi-definite matrices are called *mixed states*.

For a matrix/operator A, the *operator norm* of A is defined as

$$\|A\| = \sup\{\|Ax\| : x \in \mathbb{H}, \|x\| \leq 1\}, \qquad \|x\| = \langle x, x\rangle^{1/2}.$$

An operator A is called a *contraction* if $\|A\| \leq 1$.

One of the most important information about operators/matrices are their *spectra*. Generally, the spectrum

$\sigma(A)$ of a linear operator A acting in some Hilbert space consists of all numbers $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is not invertible. Therefore, in the finite dimensional case *the spectrum* $\sigma(A)$ of a matrix A is the set of eigenvalues of A, i.e., all numbers λ such that $Ax = \lambda x$. Eigenvalues $s_i(A)$ of the module $|A|$ are called *the singular values* (also called s-numbers) of A. For a matrix $A \in \mathbb{M}_n$, the notation $s(A) \equiv (s_1(A), s_2(A), ..., s_n(A))$ means that $s_1(A) \geq s_2(A) \geq ... \geq s_n(A)$.

Now let us recall some important norms which will be considered in this thesis.

The *Ky Fan k-norm* is the sum of all singular values, i.e.,

$$||A||_k = \sum_{i=1}^{k} s_i(A).$$

The *Schatten p-norm* is defined as

$$||A||_p = \left(\sum_{i=1}^{n} s_i^p(A) \right)^{1/p}.$$

When $p = 2$, we have the *Frobenius norm* or sometimes called the *Hilbert-Schmidt norm* :

$$\|A\|_2 = (\mathrm{Tr}\,|A|^2)^{1/2} = \left(\sum_{j=1}^{n} s_j^2(A) \right)^{1/2}.$$

Definition 1.0.1. A norm $|||\cdot|||$ on $\mathbb{M}_n$ is called *unitarily invariant* if for any matrix $A \in \mathbb{M}_n$ and for any unitary matrices $U, V \in \mathbb{M}_n$,

$$|||UAV||| = |||A|||.$$

Chapter 2

New types of operator convex functions and related inequalities

Being the most fundamental concept in convex analysis and optimization theory, the convexity of functions has been extensively studied in various contexts of pure and applied mathematics. The main aim of this chapter is to define new classes of operator convex functions based on Kubo-Ando theory of operator means even for any number of matrices [77]. More precisely, we use the family of the matrix power means to define new classes of so called *operator* (r, s)*-convex functions* and *operator* (p, h)*-convex functions*. Studying their properties, we prove some well-known inequalities for them. We also provide similar to the Hansen-Pedersen characterizations for operator (p, h)-convex and operator (r, s)-convex functions.

The results of this chapter are taken from [51] and [48].

2.1 Operator (p, h)-convex functions

Recall that let p be some positive number, J some interval in $\mathbb{R}^+$ containing the interval $[0, 1]$, and K $(\subset \mathbb{R}^+)$ a p-convex subset of $\mathbb{R}^+$ (i.e., $[\alpha x^p + (1-\alpha)y^p]^{1/p} \in K$ for all $x, y \in K$ and $\alpha \in [0, 1]$).

Definition 2.1.1. Let $h : J \to \mathbb{R}^+$ be a non-zero super-multiplicative function. A non-negative function $f : K \to \mathbb{R}$ is said to be *operator* (p, h)*-convex* (or belongs to the class $opgx(p, h, K)$) if for any $A, B \in \mathbb{M}_n^+$ with $\sigma(A), \sigma(B) \subset K$, and for $\alpha \in (0, 1)$,

$$f\left([\alpha A^p + (1-\alpha)B^p]^{1/p}\right) \le h(\alpha)f(A) + h(1-\alpha)f(B).$$

When $p = 1,\ h(\alpha) = \alpha$, we get the usual definition of operator convex functions on $\mathbb{R}^+$.

Remark 2.1.1. *An operator* (p, h)*-convex function could be either operator monotone or operator convex. However, there are many operator* (p, h)*-convex functions which are neither an operator monotone function nor an operator convex function.*

2.1.1 Some properties of operator (p, h)-convex functions

Theorem 2.1.1. *Let* $opgx(p, h, K)$ *be the class of operator* (p, h)*-convex functions. Then,*

(i) *If* $f, g \in opgx(p, h, K)$ *and* $\lambda > 0$*, then* $f + g, \lambda f \in opgx(p, h, K)$*;*

(ii) *Let h_1 and h_2 be non-negative and non-zero super-multiplicative functions defined on an interval J with $h_2 \le h_1$ in $(0,1)$. If $f \in opgx(p, h_2, K)$, then $f \in opgx(p, h_1, K)$;*

(iii) *Let $f \in opgx(p_2, h, K)$ such that f is operator monotone function on K. If $1 \le p_1 \le p_2$, then $f \in opgx(p_1, h, K)$.*

Theorem 2.1.2. *Let K be an interval in $\mathbb{R}^+$ such that $0 \in K$.*

(i) *If $f \in opgx(p, h, K)$ such that $f(0) = 0$, then*

$$f\left([\alpha A^p + \beta B^p]^{1/p}\right) \le h(\alpha)f(A) + h(\beta)f(B) \tag{2.1.6}$$

holds for arbitrary positive definite matrices A, B with spectra in K and all $\alpha, \beta \le 0$ such that $\alpha + \beta \le 1$;

(ii) *Let h be a non-negative function such that $h(\alpha) < 1/2$ for some $\alpha \in (0, 1/2)$. If f is a non-negative function satisfying (2.1.6) for all matrices A, B with spectra in K and all $\alpha, \beta \le 0$ with $\alpha + \beta \le 1$, then $f(0) = 0$.*

Corollary 2.1.1. *For $s > 0$, put $h_s(x) = x^s$ $(x > 0)$, and let $0 \in K \subset \mathbb{R}^+$. For all $f \in opgx(p, h_s, K)$, the inequality (2.1.6) holds for all $\alpha, \beta \ge 0$ with $\alpha + \beta \le 1$ if and only if $f(0) = 0$.*

2.1.2 Jensen type inequality and applications

Theorem 2.1.3. *Let h be a non-negative super-multiplicative function on J and $f \in opgx(p, h, K)$. Then for any k self-adjoint matrices A_i with spectra in K and any $\alpha_i \cdots, k)$ satisfying $\sum_{i=1}^k \alpha_i = 1$,*

$$f\left(\left[\sum_{i=1}^k \alpha_i A_i^p\right]^{1/p}\right) \le \sum_{i=1}^k h(\alpha_i) f(A_i). \tag{2.1.8}$$

Let E be a finite nonempty set of positive integers and a set of positive semi-definite matrices A_i $(i \in E)$.

$$\mathcal{F}(E) = h(W_E) f\left(\left[\frac{1}{W_E}\sum_{i\in E} w_i A_i^p\right]^{1/p}\right) - \sum_{i\in E} h(w_i) f(A_i), \tag{2.1.9}$$

where $W_E = \sum_{i\in E} w_i,\ w_i > 0$. The function $\mathcal{F}$ satisfies the triangle inequality in the following sense.

Theorem 2.1.4. *Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a super-multiplicative function, $f : K \to \mathbb{R}^+$ an operator (p, h)-convex. Let M and E be finite nonempty sets of positive integers such that $M \cap E = \emptyset$. Then for any $w_i > 0$ $(i \in M \cup E)$, and for any positive semi-definite matrices A_i $(i \in M \cup E)$ with spectra in K,*

$$\mathcal{F}(M \cup E) \le \mathcal{F}(M) + \mathcal{F}(E).$$

2.1.3 Characterizations of operator (p, h)-convex functions

Theorem 2.1.5. *Let $h : J \to \mathbb{R}^+$ be a super-multiplicative function, $f : K \to \mathbb{R}^+$ an operator (p, h)-convex function. Then for any pair of self-adjoint matrices A, B with spectra in K and for matrices C, D such that $CC^* + DD^* = I_n$,*

$$f\left([CA^pC^* + DB^pD^*]^{1/p}\right) \le 2h(1/2)\left(Cf(A)C^* + Df(B)D^*\right).$$

Theorem 2.1.6. *Let f be a non-negative function on the interval K such that $f(0) = 0$, and h a non-negative and non-zero super-multiplicative function on J satisfying $2h(1/2) \leq \alpha^{-1}h(\alpha)$ $(\alpha \in (0,1))$. Then the following statements are equivalent:*

(i) f is an operator (p,h)-convex function;

(ii) for any contraction V $(||V|| \leq 1)$ and self-adjoint matrix A with spectrum in K,

$$f\left([V^*A^pV]^{1/p}\right) \leq 2h(1/2)V^*f(A)V;$$

(iii) for any orthogonal projection Q and any self-adjoint matrix A with $\sigma(A) \subset K$,

$$f\left([QA^pQ]^{1/p}\right) \leq 2h(1/2)Qf(A)Q;$$

(iv) for any natural number k, for any families of positive operators $\{A_i\}_{i=1}^k$ in a finite dimensional Hilbert space $\mathbb{H}$ satisfying $\sum_{i=1}^k \alpha_i A_i = I_{\mathbb{H}}$ (the identity operator in $\mathbb{H}$) and for arbitrary numbers $x_i \in K$,

$$f\left(\left[\sum_{i=1}^k \alpha_i x_i^p A_i\right]^{1/p}\right) \leq \sum_{i=1}^k h(\alpha_i)f(x_i)A_i.$$

Remark 2.1.3. *Here we give an example for the function h which is different from the identify function and satisfies conditions in Theorem 2.1.6. It is easy to check that for the function $h(x) = x^3 - x^2 + x$ and for any $x, y \in [0,1]$,*

$$h(xy) - h(x)h(y) = xy(x+y)(1-x)(1-y) \geq 0.$$

Therefore, h is super-multiplicative on $[0,1]$. At the same time, the function $h(x)/x = x^2 - x + 1$ attains minimum at $x = 1/2$, and hence $2h(1/2) \leq h(x)/x$ for any $x \in (0,1)$.

2.2 Operator (r,s)-convex functions

Let r, s be positive numbers. For $X = (A_1, A_2)$ with $\sigma(A_1), \sigma(A_2) \subset K$ and $\omega_1, \omega_2 \geq 0$. Let $W := \omega_1 + \omega_2 > 0$. The weighted matrix r-power mean $M^{[r]}(X,W)$ is defined by

$$M^{[r]}(X,W) := \left(\frac{1}{W}\sum_{i=1}^2 \omega_i A_i^r\right)^{1/r}.$$

Definition 2.2.1. Let K be a r-convex subset of $\mathbb{R}^+$. A continuous function $f : K \to (0,\infty)$ is said to be *operator (r,s)-convex* if

$$f(M^{[r]}(X,W)) \leq M^{[s]}(f(X),W). \tag{2.2.16}$$

The reader may notice a similarity between this notion with the notion of (p,h)-convex functions introduced in the previous section. However there should be no confusion as h is a non-constant function.

2.2.1 Jensen and Rado type inequalities

In the following theorem we prove a Jensen type inequality for operator (r,s)-convexity. Let $A_1, ..., A_m$ be Hermitian matrices with spectra in K and $W = (\omega_1, ..., \omega_m)$ be positive numbers. We notate $X = (A_1, ..., A_m)$

and $W_m = \omega_1 + ... + \omega_m > 0$. The weighted matrix r-power mean $M^r_m(X, W)$ is defined by

$$M^{[r]}_m(X, W) := \left(\frac{1}{W_m}\sum_{i=1}^{m}\omega_i A_i^r\right)^{1/r}.$$

Theorem 2.2.1. *Let r, s be arbitrary positive numbers, and m be a natural number. If f is operator (r, s)-convex, then for $X = (A_1, \cdots, A_m)$ and $W = (\omega_1, \cdots, \omega_m)$,*

$$f(M^{[r]}_m(X, W)) \leq M^{[s]}_m(f(X), W). \tag{2.2.18}$$

When f is operator (r, s)-concave, the inequality (2.2.18) is reversed.

Also, we proves a Rado type inequality for operator (r, s)-convex functions.

Theorem 2.2.2. *Let r and s be two positive numbers and f a continuous function on K. For $m \in \mathbb{N}$, for $X = (A_1, \cdots, A_m)$ and $W = (\omega_1, \cdots, \omega_m)$, we denote*

$$a_m = W_m\left(M^{[s]}_m\left[f(X), W\right]^s - f\left(M^{[r]}_m[X, W]^s\right)\right).$$

Then, the following assertions hold:

(i) If f is operator (r, s)-convex then $\{a_m\}_{m=1}^{\infty}$ is an increasing monotone sequence;

(ii) If f is operator (r, s)-concave then $\{a_m\}_{m=1}^{\infty}$ is an decreasing monotone sequence.

2.2.2 Some equivalent conditions to operator (r, s)-convexity

Theorem 2.2.3. *Let $f : K \to \mathbb{R}^+$ be an operator (r, s)-convex function. Then for any pair of positive definite A, B with spectra in K and for matrices C, D such that $CC^* + DD^* = I$,*

$$f((CA^rC^* + DB^rD^*)^{1/r}) \leq (Cf(A)^sC^* + Df(B)^sD^*)^{1/s}. \tag{2.2.20}$$

Theorem 2.2.4. *Let f be a non-negative function on the interval K such that $f(0) = 0$. Then the following statements are equivalent:*

(i) f is an operator (r, s)-convex function;

(ii) for any contraction V ($||V|| \leq 1$) and for any positive semi-definite matrix A with spectrum in K,

$$f\left([V^*A^rV]^{1/r}\right) \leq (V^*f(A)^sV)^{1/s};$$

(iii) for any orthogonal projection Q and for any positive semi-definite matrix A with $\sigma(A) \subset K$,

$$f\left((QA^rQ)^{1/r}\right) \leq (Qf(A)^sQ)^{1/s};$$

(iv) for any natural number k and for any families of positive operators $\{A_i\}_{i=1}^k$ in a finite dimensional Hilbert space $\mathbb{H}$ such that $\sum_{i=1}^k \alpha_i A_i = I_{\mathbb{H}}$ (the identity operator in $\mathbb{H}$) and for arbitrary numbers $x_i \in K$,

$$f\left(\left[\sum_{i=1}^{k}\alpha_i x_i^r A_i\right]^{1/r}\right) \leq \left(\sum_{i=1}^{k}\alpha_i f(x_i)^s A_i\right)^{1/s}.$$

Chapter 3

Matrix inequalities and the in-sphere property

In the first section of the chapter we consider generalized reverse Cauchy inequalities for two positive definite matrices A and B and show that generalized reverse Cauchy inequalities hold under the condition $AB + BA \geq 0$. Moreover, we also show that the generalized reverse Cauchy inequality and the generalized Powers-Størmer inequality holds with respect to the unitarily invariant norms under the same condition. In the second section we prove some reverse inequalities of the matrix Heinz means and unitarily invariant norms. And the last section dedicates the in-sphere property for matrix means.

This chapter is written based on results in [50] and [52].

3.1 Generalized reverse arithmetic-geometric mean inequalities

Young inequalities for two positive matrices are important in estimating some quantum quantities, such as the quantum Chernoff bound [59] and the Tsallis relative entropy [33]. $A\sigma_f B = A^{\frac{1}{2}} f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$ is the operator mean corresponding to the function f in the sense of Kubo and Ando [61]. We call inequality

$$\frac{A+B}{2} - A\sigma_f B \leq \frac{|A-B|}{2},$$

the generalized reverse arithmetic-geometric mean (AGM) inequality.

Theorem 3.1.1 ([50]). *Let f be a strictly positive operator monotone function on $[0,\infty)$ with $f((0,\infty)) \subset (0,\infty)$ and $f(1) = 1$. Then for any positive semi-definite matrices A and B with $AB + BA \geq 0$,*

$$A + B - |A - B| \leq 2A\sigma_f B.$$

3.2 Reverse inequalities for the matrix Heinz means

It was shown in [47, Theorem 2.1] that for any operator mean σ and for any $A, B \in \mathbb{P}_n$,

$$\frac{A+B}{2} - A\sigma B \leq \frac{1}{2}A^{1/2}|I - A^{-1/2}BA^{-1/2}|A^{1/2}. \tag{3.2.9}$$

3.2.1 Reverse arithmetic-Heinz-geometric mean inequalities for unitarily invariant norms

Theorem 3.2.1. *Let $|||\cdot|||$ be an arbitrary unitarily invariant norm on $\mathbb{M}_n$. Let f be an operator monotone function on $[0,\infty)$ with $f((0,\infty)) \subset (0,\infty)$ and $f(0)=0$, and g a function on $[0,\infty)$ such that $g(t)=\frac{t}{f(t)}$ $(t \in (0,\infty))$ and $g(0)=0$. Then for any $A, B \in \mathbb{P}_n$,*

$$\begin{aligned}\left|\left|\left|\frac{A+B}{2}-\frac{1}{2}A^{1/2}\left|I-A^{-1/2}BA^{-1/2}\right|A^{1/2}\right|\right|\right| &\leq \left|\left|\left|f(A)^{1/2}g(B)f(A)^{1/2}\right|\right|\right| \\ &\leq \left|\left|\left|f(A)g(B)\right|\right|\right|.\end{aligned}$$

Corollary 3.2.1. *Let $A, B \in \mathbb{P}_n$ and $s \in [0,1]$. Then we have*

$$\left|\left|\left|\frac{A+B}{2}-\frac{1}{2}A^{1/2}|I-A^{-1/2}BA^{-1/2}|A^{1/2}\right|\right|\right| \leq \left|\left|\left|A^{1/2}B^{1/2}\right|\right|\right|.$$

Corollary 3.2.2. *Let $A, B \in \mathbb{H}_n^+$ and $s \in [0,1]$. Then we have*

$$\left|\left|\left|\frac{A+B}{2}-\frac{1}{2}A^{1/2}|I-A^{-1/2}BA^{-1/2}|A^{1/2}\right|\right|\right| \leq \frac{1}{2}\left|\left|\left|A^sB^{1-s}+A^{1-s}B^s\right|\right|\right|.$$

Corollary 3.2.3. *For any $A, B \in \mathbb{H}_n^+$ such that $AB+BA \geq 0$ and $s \in [0,1]$, we get the following inequalitites*

$$|||A+B-|A-B|||| \leq 2|||A^{1/2}B^{1/2}|||$$

and

$$|||A+B-|A-B|||| \leq 2|||A^sB^{1-s}+A^{1-s}B^s|||.$$

3.2.2 Reverse inequalities for the matrix Heinz mean with respect to Hilbert-Schmid norm

It is obvious that for any positive numbers a and b,

$$(a+b)^2-|a^2-b^2| \leq (a^sb^{1-s}+a^{1-s}b^s)^2. \tag{3.2.19}$$

Theorem 3.2.2. *For any $A, B \in \mathbb{H}_n^+$ and $X \in \mathbb{M}_n$, then*

$$||AX+XB||_2^2-||AX-XB||_2^2 \leq ||A^sXB^{1-s}+A^{1-s}XB^s||_2^2.$$

3.3 The in-sphere property for operator means

In this section we will study in-sphere property for operators. In the next theorem, we provide a new characterization of the matrix arithmetic mean by the inequality (3.1.6).

Theorem 3.3.1. *Let σ be an arbitrary symmetric mean. If for any arbitrary unitarily invariant norm $|||\cdot|||$ on $\mathbb{M}_n$,*

$$\left|\left|\left|\frac{A+B}{2}-A\sigma B\right|\right|\right| \leq \frac{1}{2}|||A-B|||$$

whenever $A, B \in \mathbb{P}_n$, then σ is the arithmetic mean.

If we replace the Kubo-Ando means by the power mean $M_p(A,B,t)=(tA^p+(1-t)B^p)^{1/p}$ with $p \in [1,2]$ then the inequality in Theorem 3.3.1 holds without the condition $AB+BA \geq 0$. In other words, the matrix power means $M_p(A,B,t)$ satisfies in-sphere property with respect to the Hilbert-Schmidt 2-norm.

Theorem 3.3.2. *Let $p \in [1,2]$, $t \in [0,1]$ and $M_p(A,B,t) = (tA^p + (1-t)B^p)^{1/p}$. Then for any pair of positive semi-definite matrices A and B,*

$$\Big|\Big|\frac{A+B}{2} - M_p(A,B,t)\Big|\Big|_2 \leq \frac{1}{2}\Big|\Big|A-B\Big|\Big|_2. \tag{3.2.28}$$

Conclusion

The thesis obtains the following results.

1. Define new class of operator (p, h)-convex functions and obtain properties for them. This is a new class of operator function, generalizing many classes of known operator functions.
2. Provide a type of Jensen inequality for operator (p, h)-convex function, generalizing for many types of Jensen inequality for known classes of operator convex functions
3. Provide a Hansen-Pedersen type inequality for operator(p, h)-convex functions, prove an inequality for index set functions for this class of functions.
4. Define a class of operator (r, s)-convex function and study some properties for them. This is also a new class of operator convex functions, generalizing the class of operator r-convex functions.
5. Prove the Jensen and Rado type inequalities for operator (r, s)-convex functions.
6. Provide some equivalent conditions for a function to be operator (p, h)-convex and (r, s)-convex, respectively.
7. Prove a generalized reverse arithmetic-geometric mean inequality involving Kubo-Ando means.
8. Prove some reverse norm inequalities for the matrix Heinz mean.
9. Obtain a new characterization of the arithmetic mean by a matrix inequality with respect to the unitarily norm.
10. Obtain ”the in-sphere property” for matrix means with respect to unitary invariant norm and Hilbert Schmidt norm. At the same time, we also show that the matrix power mean satisfies the in-sphere property with respect to the Hilbert-Schmidt norm.

Future investigation.

In the near future, we intend to continue investigation in the following direction:

1. Continue to characterize new classes of operator convexity with some well-known matrix means.

2. Let p, q be positive numbers, h be non-negative real valued super-multiplication function. We consider a general definiton as follows: A function f is called operator (p, h, q)-convex if

$$f\left([\alpha A^p + (1-\alpha)B^p]^{1/p}\right) \leq [h(\alpha)f(A)^q + h(1-\alpha)f(B)^q]^{1/q}$$

If $q = 1$ then we get the class of operator $(p, h, 1)$-convex or called operator (p, h)-convex, and if $h \equiv id$ is identity function, we get the class of operator (p, id, q)-convex functions, or called as operator (r, s)-convex functions. In the future, we intend to continue to investigate this general class of operator functions for some different cases.

3. In-sphere property of the matrix mean: We believe that the matrix power mean satisfies in-sphere property with respect to the p-Schatten norm a larger range of p and for any unitarily invariant norm.

4. Define new classes of quantum entropy in relation with new types of operator convex functions. It would be meaningful to study their properties and applications in quantum information theory.

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List of Author's Papers related to the thesis

1. D. T. Hoa, V. T. B. Khue, H. Osaka (2016), "A generalized reverse Cauchy inequality for matrices", *Linear and Multilinear Algebra*, 64, 1415-1423.

2. D. T. Hoa, V. T. B. Khue (2017), "Some inequalities for operator (p, h)-convex functions", *Linear and Multilinear Algebra*.

   http://dx.doi.org/10.1080/03081087.2017.1307914.

3. D. T. Hoa, D. T. Duc, V. T. B. Khue (2017), "A new type of operator convexity", accepted for publication in Acta Mathematica Vietnamica.

4. D. T. Hoa, V. T. B Khue, T.-Y. Tam (2017), "A new type of operator convexity", accepted for publication in Acta Mathematica Vietnamica.