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This thesis has been completed within Quy Nhon University, under the supervisor of Assoc. Prof. Dr. Phan Hoang Chon and Assoc. Prof. Dr. Nguyen Sum. I hereby assure that this research project is mine. All results are honest, have been approved by co-authors and have not been released by anyone else before.

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Pham Bich Nhu

CONTENTS

Introduction

Sigular homology and cohomology operations are the tools used to study the problem of the homotopy classification problem of topological spaces. An important issue in studying the problem of homotopy type classification of topological spaces is to determine the homotopy group, especially the stable homotopy groups of the spheres. In [1], Adams constructed a spectral sequence, known as the Adams spectral sequence, to converge on the p-adic stable homotopy group of the sphere $\pi_*^S(\mathbb{S}^0)$. The E_2 term of the Adams spectral sequence is the cohomology of the Steenrod algebra $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_p, \mathbb{F}_p)$. Since that work came into being, determining the cohomology of the Steenrod algebra has become a fascinating subject, attracting many interested and research mathematicians. Since the 60s of the last century, mathematicians have had many studies on $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_p, \mathbb{F}_p)$ for $p = 2$, typically those of Adams [1], Wang [65], May [46], Tangora [64], Lin [39], Lin-Mahowald [40], Bruner [10], and others. However, this is a very difficult problem. Until now, the problem of determining the cohomology of the Steenrod algebra remains open, especially in the case of odd p.

There are many tools and approaches to study the cohomology of the Steenrod algebra such as graded differential Lambda algebra (see Bousfield [6], Chen [11], Lin [39], Priddy [54], Singer [56], Wang [65]), May spectral sequence (see May [44], [45], Tangora [64], Lin [?], Chon-Ha [14, 15]), the minimal resolution (see Bruner [9]) and the modular invariants. Especially, the modular invariants are the algebraic transfer (also known as the Singer transfer) constructed by Singer [57] in 1989 and the Lannes-Zarati homomorphism built by Lannes-Zarati in [72].

The Lannes-Zarati homomorphism mod p was first defined by Lannes-Zarati [72] as follows, for any unstable $\mathscr{A}$-module M and for each integer $s \geq 0$,

$$\varphi_s^M : \mathrm{Ext}_{\mathscr{A}}^{s,s+t}(M, \mathbb{F}_p) \longrightarrow \mathrm{Ann}((\mathscr{R}_s M)^{\#})_t.$$

Here, for any A-module N, we denote $N^{\#}$ the linear dual of N and $\mathrm{Ann}(N^{\#})$ the subspace of $N^{\#}$ consisting of all elements annihilated by all positive degree elements of A, and $\mathscr{R}_s M$ is the Singer functor.

Moreover, the mod p Lannes-Zarati homomorphism is closely related to the mod p Hurewicz map. Indeed, if M is the reduced mod p (singular) cohomology of a pointed space X, then φ_s^M is considered as a graded associated version of the mod p Hurewicz map

$$H : \pi_*^S(X) \cong \pi_*(QX) \to H_*(QX)$$

of the infinite loop space $QX := \varinjlim \Omega^n \Sigma^n X$ in the E_2-term of Adams spectral sequence (see Lannes and Zarati [70], [71] for $p = 2$ and Kuhn [38] for odd prime p). Hence, the study the behavior of the mod p Lannes-Zarati homomorphism actually corresponds to the description of the image of the Hurewicz map.

For $p = 2$, Lannes và Zarati [72] show that $\varphi_1^{\mathbb{F}_2}$ is an isomorphism and $\varphi_2^{\mathbb{F}_2}$ is an epimorphism. Later, it is proved by Hung et. al. that $\varphi_s^{\mathbb{F}_2}$, for $3 \leq s \leq 5$, is trivial at all positive stems (see [27], [32], [34]). These results are closely related to the conjecture of Curtis [22] (and Wellington [66] for odd prime) on spherical classes through the fact that, from the results of Adams [1] and Browder [8], the Hopf invariant one elements and the Kervaire invariant one elements in $\pi_*^S\mathbb{S}^0$ (if they exist) are respectively detected by certain permanent cycles in $\mathrm{Ext}_{\mathscr{A}}^{1,*}(\mathbb{F}_2, \mathbb{F}_2)$ and $\mathrm{Ext}_{\mathscr{A}}^{2,*}(\mathbb{F}_2, \mathbb{F}_2)$. In addition, for $M = \widetilde{H}^*\mathbb{R}P^\infty$ and $M = \widetilde{H}^*\mathbb{R}P^n$, it is showed by Hung and Tuan that φ_0^M is an isomorphism, φ_1^M is an epimorphism and φ_s^M is vanishing at all positive stems for $s = 2, s = 3$ and

$s = 4$. This result also shows that the behavior of φ_s^M has a close relationship with the conjecture of Eccles (see [67] for discussion). Thus, understanding the mod p Lannes-Zarati homomorphism plays an important role in the determination of the image of the Hurewicz map as well as in the investigation of the conjectures on spherical classes.

According to above discussion, the mod 2 Lannes-Zarati homomorphism has been carefully studied by many authors for a long time while no-one undertook to investigate the mod p Lannes-Zarati homomorphism for p odd.

In this thesis, we focus on research the behavior of the mod p Lannes-Zarati homomorphism, p odd.

Specifically, we have established a chain-level representation of $(\varphi_s^M)^\#$ in Singer-Hung-Sum chain complex as well as a chain-level representation of φ_s^M in the complex $\Lambda \otimes M^\#$, for any $\mathscr{A}$-module M.

Using Lambda algebra to study the kernel and image of the mod p Lannes-Zarati homomorphism (1.6) for the case $M = \mathbb{F}_p$ can avoid using results of the "hit problem" for $\mathscr{R}_s\mathbb{F}_p$ as in [30], [25], [27], [32]. By this method, we obtain new results about the behavior of $\varphi_s^{\mathbb{F}_p}$ for $s \leq 3$ with p odd. However, for s higher, the computation remains difficult because the Adem relations of the mod p Dyer-Lashof algebra $\mathcal{R}$, considered as the dual of $\mathscr{R}_s\mathbb{F}_p$, in general, is hard to exploit.

To overcome this difficulty, we develop the power operation $\mathcal{P}^0$ acting on $\mathrm{Ext}_{\mathscr{A}}^{s,*}(\mathbb{F}_p, \mathbb{F}_p)$ (see Liulevicius [41] or May [19]). For $M = \mathbb{F}_p$ and $M = \widetilde{H}^*(B\mathbb{Z}/p)$, we show that there exist the power operations $\mathcal{P}^0$s acting on $\mathrm{Ext}_{\mathscr{A}}^{s,*}(M, \mathbb{F}_p)$ and on $(\mathbb{F}_p \otimes_{\mathscr{A}} \mathscr{R}_s M)^\#$. Moreover, these actions are compatible with each other through the mod p Lannes-Zarati homomorphism φ_s^M.

A family $\{a_i : i \geq i_0\} \subset \mathrm{Ext}_{\mathscr{A}}^{s,*}(M, \mathbb{F}_p)$ is called a $\mathcal{P}^0$-family if $a_{i+1} = \mathcal{P}^0(a_i)$ for $i \geq i_0$. The above result allows us to determine $\varphi_s^M(a_i)$ through $\varphi_s^M(a_{i_0})$, this makes reduce significantly the computation in studying the behavior of $\varphi_s^{\mathbb{F}_p}$, $s \leq 3$ and $\varphi_s^{\widetilde{H}^*(B\mathbb{Z}/p)}$, $s \leq 1$, for p odd prime. Notice that our method can use it for the case p = 2 with a little modification of degree.

In addition to the introduction, the conclusion and the references, the thesis is divided into 3 chapters.

In Chapter 1, we present the basic knowledge needed for the main part of the thesis. The new results of the thesis are presented in Chapters 2 and 3.

In Chapter 2 we study the chain-level representation of the dual of φ_s^M on Singer-Hung-Sum chain complex and the chain-level representation of the Lannes-Zarati homomorphism φ_s^M on the complex $\Lambda \otimes M^\#$.

In Chapter 3, we present the results obtained from studying the mod p Lannes-Zarati homomorphism over $\mathbb{F}_p$ and $\widetilde{H}^*(B\mathbb{Z}/p)$, including the case $p = 2$.

The main results of this thesis are published in two papers [17, 18], one preprint [50] which were reported at:

- November $9 - 10^{th}$, 2018: The Ninth Conference of the University of Natural Sciences - VNU HCMC, Ho Chi Minh City (20-minute talk) ;

- June $9 - 14^{th}$,2019: Vietnamese-American Mathematical Conference, Quy Nhon (Poster report)

- August $2^{nd} - 4^{th}$, 2019: The 3^{rd} Mathematical Conference of Central and Highland Area of Vietnam, Buon Ma Thuot (20-minute talk)

- Seminar at Mathematics Department, Quy Nhon University, Binh Dinh;

Chapter 1

Preliminaries

1.1 The Steenrod algebra

In 1947, Steenrod [61] defined

Definition 1.1.1. Let X be the topology space, the cohomology operations, $\forall i \geq 0, n \geq 0$,

$$Sq^i : H^n(X, \mathbb{F}_2) \to H^{n+i}(X, \mathbb{F}_2),$$

act naturally on the cohomology of the topological space X, called the Steenrod squaring.

In 1952, Steenrod [60] continuously construct Steenrod operations over the field $\mathbb{F}_p$ with p is odd prime.

Definition 1.1.2. Let X be the topology space,the cohomology operations, $\forall i \geq 0, n \geq 0$,

$$P^i : H^n(X, \mathbb{F}_p) \to H^{n+2(p-1)i}(X, \mathbb{F}_p),$$

act naturally on the cohomology of the topological space X, called Steenrod power.

That same year, Adem [3] proved that all relations in the Steenrod algebra are derived from the set of relationships

$$Sq^a Sq^b = \sum_{i=0}^{[a/2]} \binom{b-i-1}{a-2i} Sq^{a+b-i} Sq^i, \tag{1.1}$$

for $a < 2b$ (the case $p = 2$) and

$$P^i P^j = \sum_{t=0}^{[i/p]} (-1)^{i+t} \binom{(p-1)(j-t)-1}{i-pt} P^{i+j-t} P^t, \tag{1.2}$$

for $i < pj$,

$$\begin{aligned} P^i \beta P^j = & \sum_{t=0}^{[i/p]} (-1)^{i+t} \binom{(p-1)(j-t)}{i-pt} \beta P^{i+j-t} P^t \\ & - \sum_{t=0}^{[i-1/p]} (-1)^{i-1+t} \binom{(p-1)(j-t)-1}{i-pt-1} P^{i+j-t} \beta P^t, \end{aligned} \tag{1.3}$$

for $i \leq pj$ (the case p odd), therein binomial coefficients take follow mod p, denote $[x]$ be an integer of x, is the largest integer that does not exceed x and β is Bockstein operation.

From the above results, purely algebra, we define the algebra generated by the Steenrod operators (called Steenrod algebra) as follows.

Definition 1.1.3. The Steenrod algebra, $\mathscr{A}$, is unity, graded algebra over the field $\mathbb{F}_p$ generated by elements Sq^i of degree $i \geq 0$, satisfying $Sq^0 = 1$ and the Adem relations (1.1) (for $p = 2$); generated by elements $P^i, i \geq 0$ of degree $2(p-1)i$ and β of degree 1, satisfying $P^0 = 1, \beta^2 = 0$ and the Adem relations (1.2), (1.3) (for p odd).

Proposition 1.1.4. (See Steenrod-Epstein [61]). *The set of all admissible monomials is a basis of Steenrod's algebra $\mathscr{A}$, as graded vector space over $\mathbb{F}_p$.*

Proposition 1.1.5. (See Steenrod-Epstein [61]). *For all $i \geq 0$, operations P^k indecomposable iff k is power of p. Then, set $\{Sq^{2i} | i \geq 0\}$, for $p = 2$ and $\{P^{p^i} | i \geq 0\} \cup \{\beta\}$ for p odd, is algebraic span of $\mathscr{A}$.*

Proposition 1.1.6. (See Steenrod-Epstein [61]). *$\mathscr{A}_*$ is also a Hopf algebra and has a basis, including monomials form $\tau_0^{\varepsilon_0}\xi_1^{r_1}\tau_1^{\varepsilon_1}\xi_2^{r_2}\cdots$, where $\varepsilon_i = 0$ or 1.*

1.2 Module over the Steenrod algebra

Definition 1.2.1. A $\mathscr{A}$-module M is called unstable if for all elements $x \in M$,

- $Sq^i x = 0$ for $\deg(x) < i$ if $p = 2$.
- $\beta^\varepsilon P^i(x) = 0$ for any $\deg(x) < 2i + \varepsilon, \varepsilon = 0, 1$ if $p > 2$.

Categories of all unstable $\mathscr{A}$-modules denoted by $\mathcal{U}$.

Definition 1.2.2. Let $\mathscr{A}$-module M, the s-th suspending of M, denote $\Sigma^s M$, defined by $(\Sigma^s M)^n = M^{n-s}$. The action of the Steenrod algebra on $\sum^s M$ given by $\theta(\Sigma^s m) = (-1)^{s \deg\theta}\Sigma^s(\theta m)$ for all $m \in M$ and $\theta \in \mathscr{A}$.

1.3 The Lannes-Zarati homomorphism

The destabilization functor $\mathscr{D}\colon \mathcal{M} \to \mathcal{U}$ is the left adjoint to the inclusion $\mathcal{U} \longrightarrow \mathcal{M}$. It can be described more explicitly as follows $\mathscr{D}(M) := M/EM$, where $EM := \mathrm{Span}_{\mathbb{F}_p}\{\beta^\epsilon P^i x \colon \epsilon + 2i > \deg(x), x \in M\}$. Then, $\mathscr{D}(M)$ can be identified with the (trivial) A-submodule of M consisting of all elements in nonnegative degrees.

For any A-module M, the projection $M \longrightarrow \mathbb{F}_p \otimes_A M$ induces an A-homomorphism $\mathscr{D}(M) \longrightarrow \mathscr{D}(\mathbb{F}_p \otimes_A M)$. Thus, there exists a natural A-homomorphism $\mathscr{D}(M) \longrightarrow \mathbb{F}_p \otimes_A M$ which is the composition

$$\mathscr{D}(M) \longrightarrow \mathscr{D}(\mathbb{F}_p \otimes_A M) \hookrightarrow \mathbb{F}_p \otimes_A M.$$

This in turns induces maps between corresponding derived functors$i_s^M \colon \mathscr{D}_s(M) \longrightarrow \mathrm{Tor}_s^A(\mathbb{F}_p, M)$.

Let E_s denote an s-dimensional $\mathbb{F}_p$-vector space. It is well-known that the mod p cohomology of the classifying space BE_s is given by

$$P_s := H^*BE_s = E(x_1, \ldots, x_s) \otimes \mathbb{F}_p[y_1, \ldots, y_s],$$

where, $E(x_1, \cdots, x_s)$ and $\mathbb{F}_p[y_1, \cdots, y_s]$ are standard notations for the exterior algebra and the polynomial algebra respectively over $\mathbb{F}_p$ generated by variables $x_1, \cdots, x_s$ degree 1 and $y_1, \cdots, y_s$ degree 2.

Define $\alpha_1(M)\colon \mathscr{D}_r(\Sigma^{-1}M) \longrightarrow \mathscr{D}_{r-1}(P_1 \otimes M)$ to be the connecting homomorphism of the functor $\mathscr{D}(-)$ associated to the short exact sequence

$$0 \to P_1 \otimes M \to \hat{P} \otimes M \to \Sigma^{-1}M \to 0,$$

where $\hat{P}$ is the A-module extension of P_1 by formally adding the generator $x_1 y_1^{-1}$ in degree -1.

Put

$$\alpha_s(M) := \alpha_1(P_{s-1} \otimes M) \circ \cdots \circ \alpha_1(\ \Sigma^{-(s-1)}M),$$

then $\alpha_s(M)$ is an A-linear map from $\mathscr{D}_r(\Sigma^{-s}M)$ to $\mathscr{D}_{r-s}(P_s \otimes M)$. In particular, when $r = s$, we obtain a map $\alpha_s(M)\colon \mathscr{D}_s(\Sigma^{-s}M) \longrightarrow \mathscr{D}_0(P_s \otimes M)$.

On the other hand, for M an unstable A-module, Singer $\mathscr{R}$ construction (see Section 2.1 Chapter 2) provides a functorial A-submodule $\mathscr{R}_s M$ of $P_s \otimes M$.

Theorem 1.3.1 (Zarati [74, Theórème 2.5])**.** *For any unstable module M, the homomorphism $\alpha_s(\Sigma M) : \mathscr{D}_s(\Sigma^{1-s}M) \longrightarrow \Sigma\mathscr{R}_s M$ is an isomorphism of unstable A-modules.*

Based on the above results, for any unstable A-module M and for $s \geq 0$, there exists a homomorphism $(\bar{\varphi}_s^M)^{\#}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{D}_s(\Sigma^{1-s}M) \arrow[r, "\alpha_s(\Sigma M)"] \arrow[d, "i_s^{\Sigma^{1-s}M}"'] & \Sigma\mathscr{R}_s M \arrow[r, hook] \arrow[dl, dotted, "(\bar{\varphi}_s^M)^{\#}"] & \Sigma P_s \otimes M \\
\mathrm{Tor}_s^{\mathscr{A}}(\mathbb{F}_p, \Sigma^{1-s}M). & &
\end{tikzcd} \tag{1.4}$$

Because the Steenrod algebra $\mathscr{A}$ acts trivially on the target, $(\bar{\varphi}_s^M)^{\#}$ factors through $\mathbb{F}_p \otimes_{\mathscr{A}} \Sigma\mathscr{R}_s M$. Therefore, after suspending -1 degree, we obtain the dual of the mod p Lannes-Zarati homomorphism

Definition 1.3.2. For any unstable $\mathscr{A}$-module M and for all integers $s \geq 0$. Then, homomorphism

$$(\varphi_s^M)^{\#} : (\mathbb{F}_p \otimes_{\mathscr{A}} \mathscr{R}_s M)^t \to \mathrm{Tor}_{s,t}^{\mathscr{A}}(\mathbb{F}_p, \Sigma^{-s}M) \cong \mathrm{Tor}_{s,s+t}^{\mathscr{A}}(\mathbb{F}_p, M). \tag{1.5}$$

is called dual of modulo p Lannes-Zarati homomorphism

The linear dual

Definition 1.3.3. For any unstable $\mathscr{A}$-module M and for all integers $s \geq 0$. Then, homomorphism

$$\varphi_s^M : \mathrm{Ext}_{\mathscr{A}}^{s,s+t}(M, \mathbb{F}_p) \longrightarrow (\mathbb{F}_p \otimes_{\mathscr{A}} \mathscr{R}_s M)_t^{\#} \cong \mathrm{Ann}((\mathscr{R}_s M)^{\#})_t, \tag{1.6}$$

is the so-called modulo p Lannes-Zarati homomorphism.

1.4 The Singer-Hung-Sum chain complex

Theorem 1.4.1 (Dickson [23], Mùi [49]).

1. *The subspace of all invariants under the action of GL_s of $\mathbb{F}_p[y_1, \dots, y_s]$ is given by*

$$D[s] := \mathbb{F}_p[y_1, \dots, y_s]^{GL_s} = \mathbb{F}_p[q_{s,0}, \dots, q_{s,s-1}].$$

2. *As a $D[s]$-module, $(H^*BE_s)^{GL_s}$ is free and has a basis consisting of 1 and all elements of $\{R_{s;i_1,\dots,i_k} : 1 \le k \le s, 0 \le i_1 < \cdots < i_k \le s-1\}$.*

3. *The algebraic relations are given by*

$$R_{s;i}^2 = 0,$$
$$R_{s;i_1} \cdots R_{s;i_k} = (-1)^{k(k-1)/2} R_{s;i_1,\dots,i_k} q_{s,0}^{k-1}$$

for $0 \le i_1 < \cdots < i_k < s$.

Let $\Phi_s := H^*BE_s[L_s^{-1}]$ be the localization of H^*BE_s obtained by inverting L_s. The action of GL_s on H^*BE_s extends an action of it on Φ_s. Set $\Delta_s := \Phi_s^{T_s}, \Gamma_s := \Phi_s^{GL_s}$, where T_s is the subgroup of GL_s consisting of all upper triangle matrices with 1's on the main diagonal.

Put $u_i := M_{i;i-1}/L_{i-1}$ and $v_i := V_i/q_{i-1,0}$, then $|u_i| = 1$ and $|v_i| = 2$. From [33], we have

$$\Delta_s = E(u_1, \dots, u_s) \otimes \mathbb{F}_p[v_1^{\pm 1}, \dots, v_s^{\pm 1}],$$
$$\Gamma_s = E(R_{s;0}, \dots, R_{s;s-1}) \otimes \mathbb{F}_p[q_{s,0}^{\pm 1}, q_{s,1}, \dots, q_{s,s-1}];$$

Let Δ_s^+ be the subspace of Δ_s spanned by all monomials of the form

$$u_1^{\epsilon_1} v_1^{(p-1)i_1 - \epsilon_1} \cdots u_s^{\epsilon_s} v_s^{(p-1)i_s - \epsilon_s}, \epsilon_i \in \{0, 1\}, 1 \le i_j \le s, i_1 \ge \epsilon_1,$$

and let $\Gamma_s^+ := \Gamma_s \cap \Delta_s^+$.

From Hung-Sum [33], $\Gamma^+ := \{\Gamma_s^+\}_{s \ge 0}$ is a graded differential $\mathbb{F}_p$-module with the differential induced by

$$\partial(u_1^{\epsilon_1} v_1^{i_1} \cdots u_s^{\epsilon_s} v_s^{i_s}) = \begin{cases} (-1)^{\epsilon_1 + \cdots + \epsilon_{s-1}} u_1^{\epsilon_1} v_1^{i_1} \cdots u_{s-1}^{\epsilon_{s-1}} v_{s-1}^{i_{s-1}}, & \epsilon_s = -i_s = 1; \\ 0, & \text{otherwise,} \end{cases} \tag{1.7}$$

where $\Gamma_0^+ = \mathbb{F}_p$.

Given an A-module M, define the stable total power $S_s(x_1, y_1, \dots, x_s, y_s; m)$, for $m \in M$, as follows (see Hung-Sum [33])

$$S_s(x_1, y_1, \dots, x_s, y_s; m) := \sum_{\substack{\epsilon_j = 0,1, \\ i_j \ge 0}} (-1)^{\epsilon_1 + i_1 + \cdots + \epsilon_s + i_s} u_s^{\epsilon_s} \cdots u_1^{\epsilon_1} v_1^{-(p-1)i_1 - \epsilon_1} \cdots v_s^{-(p-1)i_s - \epsilon_s}$$
$$\otimes (\beta^{\epsilon_1} P^{i_1} \cdots \beta^{\epsilon_s} P^{i_s})(m). \tag{1.8}$$

For convenience, we put $S_s(m) := S_s(x_1, y_1, \dots, x_s, y_s; m)$, and $S_s(M) := \{S_s(m) : m \in M\} \subset \Delta_s \otimes M$.

Then Hung-Sum defined

Definition 1.4.2. For $\mathscr{A}$-module M, put $\Gamma^+M := \{(\Gamma^+M)_s\}_{s\geq 0}$, where $(\Gamma^+M)_0 := M$ and $(\Gamma^+M)_s := \Gamma_s^+S_s(M) = \{vS_s(m) : v \in \Gamma_s^+, m \in M\}$, is a differential $\mathbb{F}_p$-module and Γ^+M is called chain complex.

To remember, we call this chain complex is the Singer-Hung-Sum chain complex.

From (1.8), for any $m \in M$, $S_s(m) = \sum_I \alpha_I v^I \otimes m_I$, where $v^I \in \Delta_s, m_I \in M$ and $\alpha_I \in \mathbb{F}_p$, for any $v \in \Gamma_s^+ \subset \Delta_s, vS_s(m) = \sum_I \alpha_I vv^I \otimes m_I$.

For $v = \sum_{\epsilon,\ell} v_{\epsilon,\ell} u_s^\epsilon v_s^{(p-1)\ell-\epsilon} \in \Gamma_s^+$ and $m \in M$, where $v_{\epsilon,\ell} \in \Gamma_{s-1}^+$, the differential in Γ^+M is given by

$$\partial(vS_s(m)) = (-1)^{\deg v+1} \sum_{\epsilon,\ell} (-1)^\ell v_{\epsilon,\ell} S_{s-1}(\beta^{1-\epsilon}P^\ell m). \tag{1.9}$$

In [33], Hung-Sum showed that $H_s(\Gamma^+M) \cong \mathrm{Tor}_s^A(\mathbb{F}_p, M)$ for any A-module M. Therefore, Γ^+M is a suitable complex to compute $\mathrm{Tor}_s^A(\mathbb{F}_p, M)$.

Proposition 1.4.3. *The map $\iota^M = \{\iota_s^M\}_{s\geq 0}$ is an injection of differential $\mathbb{F}_p$-modules. Moreover, ι^M induces the isomorphism*

$$H_*(\Gamma^+M) \cong \mathrm{Tor}_*^{\mathscr{A}}(\mathbb{F}_p, M).$$

1.5 The Lambda algebra and the Dyer-Lashof algebra

In [6], Bousfield et. al. define the lambda algebra (see also Bousfield-Kan [7]), that is a differential algebra for computing the cohomology of the Steenrod algebra. Moreover, Priddy [54] proved the Lambda algebra is isomorphic to the co-Koszul resolution of the Steenrod algebra.

Purely algebraic, we can define Lambda algebra as follows.

Definition 1.5.1. The Lambda algebra, Λ, is the unital, graded, associative differential algebra over $\mathbb{F}_p$ generated by λ_{i-1} $(i > 0)$ of degree $2i(p-1)-1$ and μ_{j-1} $(j \geq 0)$ of degree $2j(p-1)$ satisfying the Adem relations (see [6], [7], [66] and [54])

$$\sum_{i+j=n} \binom{i+j}{i} \lambda_{i-1+pm}\lambda_{j-1+m} = 0,$$

$$\sum_{i+j=n} \binom{i+j}{i} (\lambda_{i-1+pm}\mu_{j-1+m} - \mu_{i-1+pm}\lambda_{j-1+m}) = 0,$$

for all $m \geq 1$ and $n \geq 0$; and

$$\sum_{i+j=n} \binom{i+j}{i} \lambda_{i+pm}\mu_{j-1+m} = 0,$$

$$\sum_{i+j=n} \binom{i+j}{i} \mu_{i+pm}\mu_{i-1+m} = 0,$$

for all $m \geq 0$ and $n \geq 0$.

The differential is given by

$$d(\lambda_{n-1}) = \sum_{i+j=n} \binom{i+j}{i} \lambda_{i-1}\lambda_{j-1},$$
$$d(\mu_{n-1}) = \sum_{i+j=n} \binom{i+j}{i} (\lambda_{i-1}\mu_{j-1} - \mu_{i-1}\lambda_{j-1}),$$
$$d(\sigma\tau) = (-1)^{\deg\sigma}\sigma d(\tau) + d(\sigma)\tau.$$

Denote $\lambda^1_{i-1} = \lambda_{i-1}$ and $\lambda^0_{i-1} = \mu_{i-1}$. Let Λ_s denote the subspace of Λ spanned by all monomial $\lambda_I = \lambda^{\epsilon_1}_{i_1-1}\cdots\lambda^{\epsilon_s}_{i_s-1}$ of length s.

Given an A-module M, the differential of complex $\Lambda \otimes M^{\#}$ is given by

$$d(\lambda \otimes h) = d(\lambda) \otimes h + \sum_{i-\epsilon\geq 0} (-1)^{\deg\lambda + (1-\epsilon)\deg h} \lambda\lambda^{\epsilon}_{i-1} \otimes h\beta^{1-\epsilon}P^i, \tag{1.10}$$

for $\lambda \in \Lambda$ and $h \in M^{\#}$.

Define the excess of λ_I or of I to be $e(\lambda_I) = e(I) = 2i_1 - \epsilon_1 - \sum_{k=2}^{s} 2(p-1)i_k + \sum_{k=2}^{s} \epsilon_s$.

Then, Curtis [22], Wellington [66] mentioned the important quotient algebra of Λ, that is the mod p Dyer-Lashof algebra $\mathcal{R}$ and this algebra is defined as follows.

Definition 1.5.2. The mod p Dyer-Lashof algebra is the quotient algebra of Λ over the (two-sided) ideal generated by all monomials of negative excess.

Let $Q^I = \beta^{\epsilon_1}Q^{i_1}\cdots\beta^{\epsilon_s}Q^{i_s}$ denote the image of λ_I under the canonical projection, and let R_s denote the subspace of R spanned by all monomials of length s, then R_s is isomorphic to $\mathscr{B}[s]^{\#}$.

1.6 Spectral sequences

Definition 1.6.1. A spectral sequences E is a family $\{E_r, d_r\}$, for $r \geq 0$ satisfies

(i) E_r is a bigraded module, where d_r is bigraded differential $(r, -r+1)$ on E_r.

(ii) For each $r \geq 0$, exist an isomorphism $H(E_r) = E_{r+1}$.

Spectral sequence of a filtered complex

A filtration F on A-module is a family of A-submodule F^pA such that $F^pA \subset F^{p+1}A$, for all integers p. If $A = \{A^s\}$ is a graded module, F have to be compatible with the graded. Let a filtration F on A, associated graded module $G(A)$ is defined by $G^p(A) = F^pA/F^{p-1}A$. If A is a graded module, associated graded module $G(A)$ is a bigraded module, is defined by $G^{p,q}(A) = F^pA^{p+q}/F^{p-1}A^{p+q}$. In this case, p is called the filtration degree, q is called the complementary degree and $p+q$ is called the total degree of an element in $G^{p,q}(A)$.

A sequence

$$\cdots \subset F^{p-1}A \subset F^pA \subset F^{p+1}A \subset \cdots$$

is an infinitely composite sequence of A and an associative graded module includes quotients of this sequence.

Filtration F is called convergent if $\cap_p F^p A = 0$ and $\cup_p F^p A = A$.

Filtration F on a chain complex C is a filter compatible with graded and differential of C (it mean $F^p C$ is a sub-complex of C include $\{F^p C^n\}$). Filtration on C induces filtration on $H^*(C)$ is defined by

$$F^p H^*(C) := \mathrm{Im}[H^*(F^p C) \to H^*(C)].$$

Chapter 2

Representation of the Lannes-Zarati homomorphism

2.1 The Singer functor

Let Σ_{p^s} be the symmetric group (of all points) of the group $E_s := (\mathbb{Z}/p)^s$ and $r_n : E_s \hookrightarrow \Sigma_{p^s}$ be the inclusion via the action by translations. Denote $\mathbb{Z}/p$ the trivial Σ_{p^s}-module of $\mathbb{Z}/p$ and $\mathcal{Z}/p$ the Σ_{p^s}-module of $\mathbb{Z}/p$ via the signature action. Put

$$\mathscr{B}[s] := \operatorname{im}\left(H^*(B\Sigma_{p^s};\mathbb{Z}/p) \xrightarrow{r_n^*} H^*(BE_s;\mathbb{Z}/p)\right);$$

$$\mathcal{B}[s] := \operatorname{im}\left(H^*(B\Sigma_{p^s};\mathcal{Z}/p) \xrightarrow{r_n^*} H^*(BE_s;r_n^*\mathcal{Z}/p)\right).$$

The structure of $\mathscr{B}[s]$ and $\mathcal{B}[s]$ are given by the following proposition.

Proposition 2.1.1 (Mui [49], Zarati [74]).

1. *$\mathscr{B}[s]$ is a free $D[s]$-module generated by*

$$\left\{1, M_{s;i_1,\ldots,i_k} L_s^{p-2+(p-1)[\frac{k-1}{2}]}\right\}$$

for $0 \leq i_1 < \cdots < i_k \leq s-1$.

2. *$\mathcal{B}[s]$ is a free $D[s]$-module generated by*

$$\left\{L_s^{\frac{p-1}{2}}, M_{s;i_1,\ldots,i_k} L_s^{\frac{p-3}{2}+(p-1)[\frac{k}{2}]}\right\}$$

for $0 \leq i_1 < \cdots < i_k \leq s-1$.

For any unstable A-module M, Zarati [74] and Powell [53] defined the (unstable) total power

$$St_s(x_1,y_1,\ldots,x_s,y_s;m),$$

for $m \in M$, is defined as follows

$$St_s(x_1,y_1,\ldots,x_s,y_s;m) := (-1)^{s\left[\frac{|m|}{2}\right]} L_s^{\frac{p-1}{2}|m|} S_s(m).$$

For convenience, we put $St_s(m) := St_s(x_1,y_1,\ldots,x_s,y_s;m)$ and $St_s(M) := \{St_s(m) : m \in M\} \subset P_s \otimes M$.

Given an unstable $\mathscr{A}$-module M, the module $\mathscr{R}_s M$ is defined by (see Zarati [74])

$$\mathscr{R}_s M = \mathscr{B}[s] \cdot St_s(M^+) \oplus \mathcal{B}[s] \cdot St_s(M^-)$$

where M^+ (resp. M^-) is the subspace consisting of all elements of even degree (resp. odd degree) of M. Then, for each $s \geq 0$, the assignment $M \rightsquigarrow \mathscr{R}_s M$ provides an exact functor from $\mathcal{U}$ to itself.

Zarati [74, Proposition 2.4.6] proved that $\mathscr{R}_s M$ is an A-submodule of $P_s^{GL_s} \otimes M \subset P_s \otimes M$. Since, we get the following proposition.

Proposition 2.1.2. *For M an unstable A-module, $\mathscr{R}_s M$ is contained in $(\Gamma^+ M)_s$. Moreover, the canonical injection $\mathscr{R}_s M \hookrightarrow (\Gamma^+ M)_s$ is given by $\lambda St_s(m) \mapsto (-1)^{sn} \lambda L_s^{\frac{p-1}{2}(2n+\delta)} S_s(m)$, for $m \in M^{2n+\delta}$ with $\delta = 0, 1$.*

Lemma 2.1.3. *For any homogeneous element γ in $\mathscr{B}[s]$ or in $\mathcal{B}[s]$, the element λ can be expressed as follows*

$$\lambda = \begin{cases} \displaystyle\sum_{I=(\epsilon_1, i_1, \ldots, \epsilon_s, i_s) \in \mathcal{I}_\gamma} \omega_I u_1^{\epsilon_1} v_1^{(p-1)i_1 - \epsilon_1} \cdots u_s^{\epsilon_s} v_s^{(p-1)i_s - \epsilon_s} & \text{if } \lambda \in \mathscr{B}[s], \\ \displaystyle\sum_{I=(\epsilon_1, i_1, \ldots, \epsilon_s, i_s) \in \mathcal{I}_\gamma} \omega_I u_1^{\epsilon_1} v_1^{(p-1)\frac{2i_1 - p^{s-1}}{2} - \epsilon_1} \cdots u_s^{\epsilon_s} v_s^{(p-1)\frac{2i_s - 1}{2} - \epsilon_s} & \text{if } \lambda \in \mathcal{B}[s], \end{cases}$$

where $\omega_I \in \mathbb{F}_p^*$ and the sum is taken over the set $\mathcal{I}_\gamma$, which is uniquely defined only depending on γ. Moreover, every string $I = (i_1, \epsilon_1, \ldots, i_s, \epsilon_s) \in \mathcal{I}_\gamma$ satisfies the condition: $\epsilon_k = 0, 1, i_k \geq \epsilon_k$ for $1 \leq k \leq s$ and

$$2i_j - \epsilon_j > \sum_{k=j+1}^{s} 2i_k(p-1) - \sum_{k=2}^{s} \epsilon_k, 1 \leq j < s, \text{ and } 2i_k \geq p^{s-k} \text{ if } \lambda \in \mathcal{B}[s].$$

Combine Proposition 2.1.2 and Lemma 2.1.3, we get the following corollary

Corollary 2.1.4. *Let $M \in \mathcal{U}$. For any homogeneous element $\gamma = \lambda St_s(m) \in \mathscr{R}_s M$ with $m \in M^{2n+\delta}$ $(\delta = 0, 1)$, γ can be expressed as follows*

$$\gamma = \sum_{I=(\epsilon_1, i_1, \ldots, \epsilon_s, i_s) \in \mathcal{I}_\gamma} \omega_I (-1)^{sn} u_1^{\epsilon_1} v_1^{(p-1)(i_1 + np^{s-1}) - \epsilon_1} \cdots u_s^{\epsilon_s} v_s^{(p-1)(i_s + n) - \epsilon_s} S_s(m),$$

where $\omega_I \in \mathbb{F}_p^*$ and the sum is taken over the set $\mathcal{I}_\gamma$, which is uniquely defined only depending on γ. Moreover, every string $I = (i_1, \epsilon_1, \ldots, i_s, \epsilon_s) \in \mathcal{I}_\gamma$ satisfies the condition: $\epsilon_k = 0, 1, i_k \geq \epsilon_k$ for $1 \leq k \leq s$ and

$$2i_j - \epsilon_j > \sum_{k=j+1}^{s} 2i_k(p-1) - \sum_{k=2}^{s} \epsilon_k, 1 \leq j < s, \text{ and } 2i_k \geq p^{s-k} \text{ if } \delta = 1. \tag{2.1}$$

Using Corollary 2.1.4 and formula (1.9), we obtain the following corollary

Corollary 2.1.5. *For an unstable A-module M, the canonical inclusion $\mathscr{R}_s M \hookrightarrow (\Gamma^+ M)_s$ maps to cycles in $(\Gamma^+ M)_s$.*

Proposition 2.1.6. *Given M an unstable A-module, $\mathscr{R}_sM$ has an $\mathbb{F}_p$-basis given by*

$$\mathscr{C} := \left\{ R_{s;0}^{\sigma_1} q_{s,0}^{j_1} \cdots R_{s;s-1}^{\sigma_s} q_{s,s-1}^{j_s} S_s(m) \right\}$$

for m running through a homogeneous basis of M, $\sigma_k \in \{0,1\}, j_1 \in \mathbb{Z}$, $j_k \geq 0, 2 \leq k \leq s$ and $2j_1 + \sigma_1 + \cdots + \sigma_s \geq |m|$.

Lemma 2.1.7. *The map $\phi_n : \mathscr{J}_n \longrightarrow \mathscr{I}_n$ given by $\phi_n(\sigma_1, j_1, \ldots, \sigma_s, j_s) = (\epsilon_1, i_1, \ldots, \epsilon_s, i_s)$ where $\epsilon_k = \sigma_k$ and $i_k = p^{s-k}(j_1 + \sigma_1 + \cdots + j_k + \sigma_k) + \sum_{t=0}^{s-k-1}(p^{s-k} - p^t)(j_{k+t+1} + \sigma_{k+t+1})$, for $1 \leq k \leq s$, is a bijection.*

Lemma 2.1.8. *Given an unstable A-module M, for any homogeneous element $m \in M$, let $(\sigma_1, j_1, \ldots, \sigma_s, j_s) \in \mathscr{J}_{|m|}$ and $(\epsilon_1, i_1, \ldots, \epsilon_s, i_s) = \phi_{|m|}(\sigma_1, j_1, \ldots, \sigma_s, j_s)$. Then,*

$$R_{s;0}^{\sigma_1} q_{s,0}^{j_1} \cdots R_{s;s-1}^{\sigma_s} q_{s,s-1}^{j_s} S_s(m) = u_1^{\epsilon_1} v_1^{(p-1)i_s - \epsilon_1} \cdots u_s^{\epsilon_s} v_s^{(p-1)i_s - \epsilon_s} S_s(m) + \textit{smaller monomials.}$$

Proposition 2.1.9. *Given an unstable A-module M, the set of all elements*

$$Q^I \otimes \ell = \beta^{\epsilon_s} Q^{i_1} \cdots \beta^{\epsilon_s} Q^{i_s} \otimes \ell,$$

for $I \in \mathscr{I}_{|\ell|}$ and ℓ running through a homogeneous basis of $M^\#$, represents an $\mathbb{F}_p$-basis of $(\mathscr{R}_sM)^\#$.

2.2 The chain-level representation of the Lannes-Zarati homomorphism

In order to investigate the behavior of the mod p Lannes-Zarati homomorphism, we first construct a chain-level representation of $(\varphi_s^M)^\#$ in the Singer-Hung-Sum chain complex as well as a chain-level representation of φ_s in the Lambda algebra.

The chain-level representation of $(\varphi_s^M)^\#$ in the Singer-Hung-Sum chain complex is given by the following theorem.

Theorem 2.2.1 (Chon-Nhu [17, Theorem 3.1])**.** *The inclusion $(\widetilde{\varphi}_s^M)^\# : \mathscr{R}_sM \longrightarrow (\Gamma^+M)_s$ given by*

$$\gamma \mapsto (-1)^{\frac{(s-2)(s-1)}{2}} \gamma$$

is a chain-level representation of the dual of the Lannes-Zarati homomorphism $(\varphi_s^M)^\#$.

Proposition 2.2.2 (Chon-Nhu [17, Prposition 3.2])**.** *The map*

$$\psi_s^{\Sigma M} : \Sigma\mathscr{R}_sM \longrightarrow \mathscr{D}B_s(\mathscr{A}, \mathscr{A}, \Sigma^{1-s}M)$$

given by

$$\Sigma\gamma \mapsto (-1)^{\frac{(s-2)(s-1)}{2} + (s-1)(\deg\gamma + \delta)}[X(\widetilde{\gamma})]$$

is a chain-level representation of the homomorphism

$$(\alpha_s(\Sigma M))^{-1} : \Sigma\mathscr{R}_sM \longrightarrow \mathscr{D}_s(\Sigma^{1-s}M).$$

For $M = \mathbb{F}_p$, from Zarati [74], $\mathscr{R}_s\mathbb{F}_p \cong \mathscr{B}[s]$. Therefore, we obtain the following corollary

Corollary 2.2.3 (Chon-Nhu [17, Corollary 3.3]). *The inclusion* $(\widetilde{\varphi}_s^{\mathbb{F}_p})^{\#} : \mathscr{R}_s\mathbb{F}_p \cong \mathscr{B}[s] \longrightarrow \Gamma_s^+$ *given by*

$$\gamma \mapsto (-1)^{\frac{(s-2)(s-1)}{2}}\gamma$$

is a chain-level representation of the dual of the Lannes-Zarati homomorphism $(\varphi_s^{\mathbb{F}_p})^{\#}$.

Hence, by taking dual Corollary 2.2.3, we have the following proposition, which is our main tool for studying the behavior of the mod p Lannes-Zarati homomorphism.

Proposition 2.2.4 (Chon-Nhu [17, Proposition 3.4]). *The projection* $\widetilde{\varphi}_s^{\mathbb{F}_p} : \Lambda_s \longrightarrow R_s$ *given by*

$$\widetilde{\varphi}_s(\lambda_I) = (-1)^{\frac{(s-2)(s-1)}{2}}Q^I$$

is a chain-level representation of the Lannes-Zarati homomorphism $\varphi_s^{\mathbb{F}_p}$.

Proposition 2.2.5 (Chon-Nhu [18, Proposition 3.7]). *For any unstable A-module M, the projection*

$$\widetilde{\varphi}_s^M : \Lambda_s \otimes M^{\#} \longrightarrow (\mathscr{R}_sM)^{\#}$$

given by

$$\lambda_I \otimes \ell \longrightarrow (-1)^{\frac{(s-1)(s-2)}{2}}[Q^I \otimes \ell]$$

is a chain-level representation of the mod p Lannes-Zarati homomorphism φ_s^M.

2.3 The proof of Proposition 2.2.2

For any $M \in \mathcal{U}$, from the short exact sequence $0 \to \Sigma^{2-s}P_1 \otimes M \to \Sigma^{2-s}\hat{P} \otimes M \to \Sigma^{1-s}M \to 0$, we have the connecting homomorphism
$\delta_1(\Sigma^{i+2-s}P_i \otimes M) : H_*(EB_*(\mathscr{A}, \mathscr{A}, \Sigma^{i+1-s}P_i \otimes M)) \longrightarrow H_{*-1}(EB_*(\mathscr{A}, \mathscr{A}, \Sigma^{i+2-s}P_{i+1} \otimes M))$,
for $0 \leq i \leq s-2$, where $P_0 = \mathbb{F}_p$,

For any A-module N, from the definition of the functor $\mathscr{D}$, one gets the short exact sequence of chain complexes $0 \longrightarrow EB(\mathscr{A}, \mathscr{A}, N) \longrightarrow B(\mathscr{A}, \mathscr{A}, N) \longrightarrow \mathscr{D}(B(\mathscr{A}, \mathscr{A}, N)) \longrightarrow 0$.

Because $B(\mathscr{A}, \mathscr{A}, N)$ is acyclic, for $s \geq 1$, the connecting homomorphism

$$\partial_* : H_s(\mathscr{D}(B(\mathscr{A}, \mathscr{A}, N)) \xrightarrow{\cong} H_{s-1}(EB(\mathscr{A}, \mathscr{A}, N)) \tag{2.2}$$

is isomorphic.

Letting $N = \Sigma^{1-s}M$, one gets the commutative diagram

$$\begin{array}{ccccccc}
\mathscr{D}_s(\Sigma^{1-s}M) & \xrightarrow{\alpha_1} & \mathscr{D}_{s-1}(\Sigma^{2-s}P_1 \otimes M) & \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_1} & \mathscr{D}_1(P_{s-1} \otimes M) \\
\downarrow{\scriptstyle \partial_*} & & \downarrow{\scriptstyle \partial_*} & & \downarrow{\scriptstyle \partial_*} \\
H_{s-1}(EB_*(\Sigma^{1-s}M)) & \xrightarrow[\delta_1]{} & H_{s-2}(EB_*(\Sigma^{2-s}P_1 \otimes M)) & \xrightarrow[\delta_1]{} \cdots \xrightarrow[\delta_1]{} & H_0(EB_*(P_{s-1} \otimes M)),
\end{array}$$

where, for convenience, in the diagram, $B_*(\mathscr{A}, \mathscr{A}, N)$ is shortly denoted by $B_*(N)$.

From the diagram, the homomorphism $\alpha_s(\Sigma M)$ can be computed by

$$\alpha_s(\Sigma M) = \alpha_1(\Sigma P_{s-1} \otimes M) \circ \partial_*^{-1} \circ \delta_1(P_{s-2} \otimes M) \circ \cdots \circ \delta_1(\Sigma^{2-s}M) \circ \partial_*.$$

Put $\delta_{s-1} := \alpha_1(\Sigma P_{s-1} \otimes M) \circ \partial_*^{-1} \circ \delta_1(P_{s-2} \otimes M) \circ \cdots \circ \delta_1(\Sigma^{2-s}M)$, then $\alpha_s(\Sigma M) = \delta_{s-1} \circ \partial_*$.

Lemma 2.3.1. *The element* $\widetilde{\gamma} \in EB_{s-1}(\mathscr{A}, \mathscr{A}, \Sigma^{1-s}M) \subset B_{s-1}(\mathscr{A}, \mathscr{A}, \Sigma^{1-s}M)$.

Lemma 2.3.2. *The element* $[X(\widetilde{\gamma})]$ *is a cycle in* $\mathscr{D}B_s(\mathscr{A}, \mathscr{A}, \Sigma^{1-s}M)$. *Moreover, under the homomorphism* $\partial_* : \mathscr{D}_s(\Sigma^{1-s}M) \xrightarrow{\cong} H_{s-1}(EB_{s-1}(\mathscr{A}, \mathscr{A}, \Sigma^{1-s}M)), \partial_*([X(\widetilde{\gamma})]) = [\widetilde{\gamma}]$.

Lemma 2.3.3. *The image of* $[\widetilde{\gamma}] \in H_{s-1}(EB_{s-1}(\mathscr{A}, \mathscr{A}, \Sigma^{1-s}M))$ *under* δ_{s-1} *is given by*

$$\delta_{s-1}([\widetilde{\gamma}]) = (-1)^{\frac{(s-2)(s-1)}{2}+(s-1)(\deg\gamma+\delta)}[\Sigma\gamma] \in \mathscr{D}_0(\Sigma P_s \otimes M).$$

Lemma 2.3.4. *Let* $\gamma = \sum_{I\in\mathcal{I}_\gamma} \omega_I (-1)^{sn} u_1^{\epsilon_1} v_1^{(p-1)(i_1+np^{s-1})-\epsilon_1} \cdots u_s^{\epsilon_s} v_s^{(p-1)(i_s+n)-\epsilon_s} S_s(m) \in \mathscr{R}_s M$, *where* $m \in M^{2n+\delta}$ $(\delta = 0, 1)$, *then*

$$\gamma = \sum_{I\in\mathcal{I}_\gamma} \omega_I (-1)^{i_1+\cdots+i_s} \times \beta^{1-\epsilon_1} P^{i_1+np^{s-1}}(x_1 y_1^{-1}(\beta^{1-\epsilon_2} P^{i_2+np^{s-2}}(x_2 y_2^{-1} \cdots (\beta^{1-\epsilon_s} P^{i_s+n}(x_s y_s^{-1} \otimes m))))).$$

In order to prove Lemma 2.3.4, we require the following results

Lemma 2.3.5 (Hung-Sum [33]). *Let* $H^*BE_1 = E(x) \otimes \mathbb{F}_p[y]$, *and let* $m, n \in M$ *for any* $\mathscr{A}$*-algebra* M . *Then, we have*

1. $S_s(mn) = S_s(m) \cdot S_s(n)$;
2. $S_s(x) = (-1)^s u_{s+1}$;
3. $S_s(y) = (-1)^s v_{s+1}$.

Corollary 2.3.6.

1. $S_1(u_i) = -u_{i+1}$;
2. $S_1(v_i) = -v_{i+1}$.

2.4 The power operations

In the early 60s, Liulevicius [41], [42] showed the existence of the squaring operations

$$Sq^0 : \mathrm{Ext}_{\mathscr{A}}^{s,s+t}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{s+i,2(s+t)}(\mathbb{F}_2, \mathbb{F}_2).$$

The operations Sq^0 is called the classical squaring operation.

Hung [26] constructed the squaring operator on the duality of the Dickson algebra $P(\mathbb{F}_2 \otimes_{GL_s} H_*(B\mathbb{E}_s))$

$$Sq^0 : P(\mathbb{F}_2 \otimes_{GL_s} H_*(B\mathbb{E}_s))_d \longrightarrow P(\mathbb{F}_2 \otimes_{GL_s} H_*(B\mathbb{E}_s))_{(2d+s)}.$$

Then, Hung [27, Theorem 1.3] showed that squaring operation Sq^0 on $P(\mathbb{F}_2 \otimes_{GL_s} H_*(B\mathbb{E}_s))$ that commutates with the classical squaring operation Sq^0 on $\mathrm{Ext}_{\mathscr{A}}^{s,s+t}(\mathbb{F}_2, \mathbb{F}_2)$ through the mod 2 Lannes-Zarati homomorphism $\varphi_s^{\mathbb{F}_2}$. This operation is developed by Hung-Tuan in [34].

From Liulevicius [41], [42] and May [46], there exists the power operation (p odd)

$$\mathcal{P}^0 : \mathrm{Ext}_A^{s,s+t}(\mathbb{F}_p, \mathbb{F}_p) \longrightarrow \mathrm{Ext}_A^{s,p(s+t)}(\mathbb{F}_p, \mathbb{F}_p).$$

Its chain-level representation in Λ is given by

$$\widetilde{\mathcal{P}}^0(\lambda_{i_1-1}^{\epsilon_1} \cdots \lambda_{i_s-1}^{\epsilon_s}) = \begin{cases} \lambda_{pi_1-1}^{\epsilon_1} \cdots \lambda_{pi_s-1}^{\epsilon_s}, & \epsilon_1 = \cdots = \epsilon_s = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.4.1. *The operation* $\widetilde{\mathcal{P}}^0$ *does not make increasing the excess of elements in* Λ*, therefore, the exists an operation, which is also denoted by* $\widetilde{\mathcal{P}}^0$*, acting on the Dyer-Lashof algebra* R *given by*

$$\widetilde{\mathcal{P}}^0(\beta^{\epsilon_1}Q^{i_1}\cdots\beta^{\epsilon_s}Q^{i_s}) = \begin{cases} \beta^{\epsilon_1}Q^{pi_1}\cdots\beta^{\epsilon_s}Q^{pi_s}, & \epsilon_1=\cdots=\epsilon_s=1,\\ 0, & \textit{otherwise.}\end{cases}$$

Lemma 2.4.2. *The operation* $\widetilde{\mathcal{P}}^0$ *is compatible with the action of* A*. In particular,*

$$\widetilde{\mathcal{P}}^0((\beta^{\epsilon_1}Q^{i_1}\cdots\beta^{\epsilon_s}Q^{i_s})P^k) = (\widetilde{\mathcal{P}}^0(\beta^{\epsilon_1}Q^{i_1}\cdots\beta^{\epsilon_s}Q^{i_s}))P^{pk}. \tag{2.3}$$

Similarly, the power operation $\widetilde{\mathcal{P}}^0$ acting on $\Lambda\otimes H$ also induces a power operation on $(\mathbb{F}_p\otimes_A \mathscr{R}_sP)^{\#}$ which is also denoted by $\mathcal{P}^0$.

Proposition 2.4.3. *The following diagram is commutative*

$$\begin{array}{ccc} \mathrm{Ext}_A^{s,s+t}(M,\mathbb{F}_p) & \xrightarrow{\mathcal{P}^0} & \mathrm{Ext}_A^{s,p(s+t)}(M,\mathbb{F}_p) \\ \varphi_s^M\big\downarrow & & \varphi_s^M\big\downarrow \\ (\mathbb{F}_p\otimes_A\mathscr{R}_sM)_t^{\#} & \xrightarrow{\mathcal{P}^0} & (\mathbb{F}_p\otimes_A\mathscr{R}_sM)_{p(s+t)-s}^{\#}, \end{array}$$

for $M=\mathbb{F}_p$ *and* $M=P$.

2.5 The case $p=2$

Proposition 2.5.1. *Given an unstable* A*-module* M*, the set of all elements* $Q^I\otimes\ell$ *for* ℓ *running through a homogeneous basis of* $M^{\#}$*,* I *admissible and* $exc(I)\geq|\ell|$*, represents an* $\mathbb{F}_2$*-basis of* $(\mathscr{R}_sM)^{\#}$*.*

Proposition 2.5.2. *For an unstable* A*-module* M*, the projection* $\widetilde{\varphi}_s^M:\Lambda_s\otimes M^{\#}\longrightarrow(\mathscr{R}_sM)^{\#}$ *given by*

$$\lambda_I\otimes\ell\longrightarrow[Q^I\otimes\ell]$$

is a chain-level representation of the mod 2 Lannes-Zarati homomorphism φ_s^M*.*

Proposition 2.5.3 (Hung-Tuan [34, Theorem 4.1])**.** *The following diagram is commutative*

$$\begin{array}{ccc} \mathrm{Ext}_A^{s,s+t}(M,\mathbb{F}_2) & \xrightarrow{Sq^0} & \mathrm{Ext}_A^{s,2(s+t)}(M,\mathbb{F}_2) \\ \varphi_s^M\big\downarrow & & \varphi_s^M\big\downarrow \\ (\mathbb{F}_2\otimes_A\mathscr{R}_sM)_t^{\#} & \xrightarrow{Sq^0} & (\mathbb{F}_2\otimes_A\mathscr{R}_sM)_{2t+s}^{\#}, \end{array}$$

for $M=\mathbb{F}_2$ *and* $M=\widetilde{H}^*(B\mathbb{Z}/2)$*.*

Chapter 3

The image of the Lannes-Zarati homomorphism

3.1 The image of the mod p Lannes-Zarati homomorphism on $\mathbb{F}_p$

Theorem 3.1.1 (Chon-Nhu [17, Theorem 4.1]). *The first Lannes-Zarati homomorphism*

$$\varphi_1 : \mathrm{Ext}_A^{1,1+t}(\mathbb{F}_p, \mathbb{F}_p) \longrightarrow \mathrm{Ann}(\mathscr{B}[1]^{\#})_t$$

is an isomorphism.

Theorem 3.1.2 (Chon-Nhu [17, Theorem 4.2]). *The second Lannes-Zarati homomorphism*

$$\varphi_2 : \mathrm{Ext}_A^{2,2+t}(\mathbb{F}_p, \mathbb{F}_p) \longrightarrow \mathrm{Ann}(\mathscr{B}[2]^{\#})_t$$

is only non-trivial at the stems $t = 0$ and $t = 2(p-1)p^{i+1} - 2, i \geq 0$.

Remark 3.1.3 (Chon-Nhu [17, Remark 4.3]). From the result of Wellington [**?**, Theorem 11.11], $\mathrm{Ann}(R_2)$ is spanned by $Q^0Q^0, \beta Q^{p^i(p-1)}\beta Q^{p^i}, i \geq 0$, and $Q^{s(p-1)}Q^s, s = p^i + \cdots + 1, i > 0$. Therefore, φ_2 is not an epimorphism.

Theorem 3.1.4 (Chon-Nhu [18, Theorem 5.1]). *The third Lannes-Zarati homomorphism*

$$\varphi_3^{\mathbb{F}_p} : \mathrm{Ext}_A^{3,3+t}(\mathbb{F}_p, \mathbb{F}_p) \longrightarrow (\mathbb{F}_p \otimes_A \mathscr{R}_3\mathbb{F}_p)_t^{\#}$$

is a monomorphism for $t = 0$ and vanishing for all $t > 0$.

Lemma 3.1.5. *If $\lambda_I \in \Lambda_s$ and $\lambda_J \in \Lambda_\ell$ such that $\widetilde{\varphi}_s^{\mathbb{F}_p}(\lambda_I) = 0$ or $\widetilde{\varphi}_\ell^{\mathbb{F}_p}(\lambda_J) = 0$ then $\widetilde{\varphi}_{s+\ell}^{\mathbb{F}_p}(\lambda_I\lambda_J) = 0$.*

3.2 The cohomology of the Steenrod algebra

In this section, we construct a spectral sequence over the complex $\Lambda \otimes \widetilde{H}_*(B\mathbb{Z}/p)$, which is a generalized version of one used in Cohen-Lin-Mahowld [20], Lin [39] and Chen [11]. After that, we use this spectral sequence to compute the cohomology of Steenrod algebra $\mathrm{Ext}_A^{s,s+t}(\widetilde{H}^*(B\mathbb{Z}/p), \mathbb{F}_p)$, and use this result in order to investigate the behavior of the Lannes-Zarati $\varphi_s^{\widetilde{H}^*(B\mathbb{Z}/p)}$.

Spectral sequences on complex chain $\Lambda\otimes \widetilde{H}_*(B\mathbb{Z}/p)$

Let $F^n := F^n(\Lambda\otimes \widetilde{H}_*(B\mathbb{Z}/p)), n\geq 0$ is filter on complex chain $\Lambda\otimes \widetilde{H}_*(B\mathbb{Z}/p)$, where $F^0(\Lambda\otimes \widetilde{H}_*(B\mathbb{Z}/p)) := 0$ and with $n>0$,

$$F^n(\Lambda\otimes \widetilde{H}_*(B\mathbb{Z}/p)) := \{\lambda\otimes h\in \Lambda\otimes \widetilde{H}_*(B\mathbb{Z}/p) : |h|\leq n\}.$$

Then, $E_0^{n,s,t} = (F^n(\Lambda_s\otimes\widetilde{H}_*(B\mathbb{Z}/p))/F^{n-1}(\Lambda_s\otimes\widetilde{H}_*(B\mathbb{Z}/p)))^t \cong \Sigma^n\Lambda_s$, therefore, $E_1^{n,s,t} = H^*(E_0^{n,s,t}) \cong \Sigma^n \operatorname{Ext}^{s,s+t-n}_{\mathscr{A}}(\mathbb{F}_p,\mathbb{F}_p)$, and $E_\infty^{n,s,t} \cong (F^nH^s(\Lambda\otimes\widetilde{H}_*(B\mathbb{Z}/p))/F^{n-1}H^s(\Lambda\otimes\widetilde{H}_*(B\mathbb{Z}/p)))^t$, where $F^nH^s(\Lambda\otimes \widetilde{H}_*(B\mathbb{Z}/p)) := \operatorname{im}\left(H^s(F^n(\Lambda\otimes\widetilde{H}_*(B\mathbb{Z}/p)))\longrightarrow H^s(\Lambda\otimes\widetilde{H}_*(B\mathbb{Z}/p))\right)$.

Thus, $\oplus_{n\geq 1}E_\infty^{n,s,t}\cong \operatorname{Ext}^{s,s+t}_{\mathscr{A}}(\widetilde{H}^*(B\mathbb{Z}/p),\mathbb{F}_p)$.

The cohomology of Steenrod algebra

The differential $E_r^{*,0,*}\xrightarrow{d_r} E_r^{*,1,*}$ is given by the following lemma.

Lemma 3.2.1. *The non-trivial differentials $E_r^{*,0,*}\xrightarrow{d_r} E_r^{*,1,*}$ are listed as follows:*

(3.2.1.1) $b^{[t]}\longrightarrow \alpha_0 ab^{[t-1]}$, *for* $t\geq 1$;

(3.2.1.2) $ab^{[(mp+k)p^i-1]}\longrightarrow -(k+1)h_iab^{[((m-1)p+k+1)p^i-1]}$, *for* $i\geq 0, 1\leq k\leq p-1$, $m\geq 1$.

Theorem 3.2.2 (Chon-Nhu [18, Theorem 5.3], Crossley [21, Theorem 1.1])**.** *The Ext group $\operatorname{Ext}^{0,t}_A(\widetilde{H}^*(B\mathbb{Z}/p),\mathbb{F}_p)$ has an $\mathbb{F}_p$-basis consisting of all elements*

1. $\widehat{h}_i := \left[ab^{[(p-1)p^i-1]}\right]\in \operatorname{Ext}^{0,2(p-1)p^i-1}_A(\widetilde{H}^*(B\mathbb{Z}/p),\mathbb{F}_p), i\geq 0$;
2. $\widehat{h}_i(k) := \left[ab^{[kp^i-1]}\right]\in \operatorname{Ext}^{0,2kp^i-1}_A(\widetilde{H}^*(B\mathbb{Z}/p),\mathbb{F}_p), i\geq 0, 1\leq k< p-1$.

The differential $E_r^{*,1,*}\xrightarrow{d_r} E_r^{*,2,*}$ is given by the following lemma.

Lemma 3.2.3. *The non-trivial differentials $E_r^{*,1,*}\xrightarrow{d_r} E_r^{*,2,*}$ are listed as follows:*

(3.2.3.1) $\alpha_0b^{[t]}\longrightarrow \alpha_0^2ab^{[t-1]}$, *for* $t\geq 1$;

(3.2.3.2) $\alpha_0ab^{[mp+k]}\longrightarrow -\binom{k+2}{2}\rho ab^{[(m-2)p+k+2]}$, *for* $0\leq k<p-2, m\geq 2$;

(3.2.3.3) $\alpha_0ab^{[(mp+k)p^i-1]}\longrightarrow (k+1)\alpha_0h_iab^{[((m-1)p+k+1)p^i-1]}$, *for* $i\geq 1, m\geq 1$;

(3.2.3.4) $\alpha_0ab^{[(mp+k)p^i-p+p-2]}\longrightarrow (k+1)\alpha_0h_iab^{[((m-1)p+k+1)p^i-p+p-2]}$, *for* $i\geq 1, m\geq 1$;

(3.2.3.5) $h_ib^{[t]}\longrightarrow -h_i\alpha_0ab^{[t-1]}$, *for* $i\geq 1, t\geq 1$;

(3.2.3.6) $h_0b^{[mp+\ell]}\longrightarrow \frac{1}{2}(\ell-1)\rho ab^{[(m-1)p+\ell]}$, *for* $m\geq 1$ *và* $\ell\neq 1$;

(3.2.3.7) $h_0b^{[(mp+e)p^i-p^2+kp+1]}\longrightarrow -(e+1)h_0h_ib^{[((m-1)p+e+1)p^i-p^2+kp+1]}$, *for* $i\geq 2, m\geq 0$;

(3.2.3.8) $h_iab^{[(mp+k)p^j-1]}\longrightarrow -(k+1)h_ih_jab^{[((m-1)p+k+1)p^j-1]}$, *for* $i\geq 1, m\geq 1, 0\leq j<i$;

(3.2.3.9) $h_iab^{[(mp+k)p^{i-1}+p^{i-1}-1]}\longrightarrow -\binom{k+2}{2}h_{i-1;1,2}ab^{[((m-2)p+k+2)p^{i-1}+p^{i-1}-1]}$, *for* $i\geq 1, m\geq 1$;

(3.2.3.10) $h_iab^{[(mp+k)p^i+(p-1)p^{i-1}-1]}\longrightarrow -\frac{1}{2}(k-1)h_{i-1;2,1}ab^{[((m-1)p+k)p^i+p^i-1]}$, *for* $i\geq 1, m\geq 1$;

(3.2.3.11) $h_iab^{[(mp+k)p^{i+2}+rp^{i+1}+p^i+u]}\longrightarrow\binom{r+2}{2}h_{i;2,1}ab^{[(mp+k-2)p^{i+2}+(r+2)p^{i+1}+p^i+u]}$, *for* $i\geq 1, m\geq 0, u=(p-1)p^{i-1}-1$;

(3.2.3.12) $h_iab^{[(mp+k)p^j-p^{i+2}+v]} \longrightarrow -(k+1)h_ih_jab^{[((m-1)p+k+1)p^j-p^{i+2}+v]}$, *for* $j-2 \geq i \geq 1, m \geq 1, v=(p-2)p^{i+1}+p^i+(p-1)p^{i-1}-1$;

(3.2.3.13) $h_iab^{[(mp+k)p^j-p^{i+1}+p^i+u]} \longrightarrow -(k+1)h_ih_jab^{[((m-1)p+k+1)p^j-p^{i+1}+p^i+u]}$, *for* $j-2 \geq i \geq 1, m \geq 0, u=(p-1)p^{i-1}-1$;

(3.2.3.14) $h_iab^{[(mp+k)p^{i+1}+p^i-1]} \longrightarrow \tilde{\lambda}_iab^{[((m-1)p+k+2)p^{i+1}-1]}$, *for* $i \geq 0, m \geq 1$;

(3.2.3.15) $h_iab^{[mp^{i+2}+kp^{i+1}+ep^i+p^i-1]} \longrightarrow \frac{(k+2)(e+1)}{2}h_{i;1,2}ab^{[(m-1)p^{i+2}+kp^{i+1}+(e+1)p^i+p^i-1]}$, *for* $i \geq 0, m \geq 0$;

(3.2.3.16) $h_iab^{[(mp+\ell)p^j-p^{i+2}+w]} \longrightarrow -(\ell+1)h_ih_jab^{[((m-1)p+\ell+1)p^j-p^{i+2}+w]}$, *for* $j-2 \geq i \geq 0, m \geq 1, w=(p-2)p^{i+1}+ep^i+p^i-1$;

(3.2.3.17) $h_iab^{[(mp+\ell)p^j-p^{i+2}+kp^{i+1}+x]} \longrightarrow -(\ell+1)h_ih_jab^{[((m-1)p+\ell+1)p^j-p^{i+2}+kp^{i+1}+x]}$, *for* $j-2 \geq i \geq 0, m \geq 1, k \neq p-2, x=p^{i+1}-1$.

Proposition 3.2.4 (Chon-Nhu [18, Proposition A.3])**.** *The infinite term $E_\infty^{*,1,*}$ has a $\mathbb{F}_p$-basis consisting of all elements given in Table 3.1.*

Table 3.1: The generators of $E_\infty^{*,1,*}$

Elements	Represented by	t	Range of indexes
$\alpha_0\widehat{h}_i$	$\alpha_0ab^{[(p-1)p^i-1]}$	$2(p-1)p^i-1$	$i \geq 1$
$\alpha_0\widehat{h}_i(k)$	$\alpha_0ab^{[kp^i-1]}$	$2kp^i-1$	$i \geq 1, 1 \leq k < p-1$
$\widehat{\alpha}(\ell)$	$\alpha_0ab^{[p+\ell]}$	$2(p+\ell)+1$	$0 \leq \ell < p-2$
$h_i\widehat{h}_i(1)$	$h_iab^{[p^i-1]}$	$2(p-1)p^i+2p^i-2$	$i \geq 0$
$h_i\widehat{h}_j$	$h_iab^{[(p-1)p^j-1]}$	$2(p-1)(p^i+p^j)-2$	$0 \leq j,i; j \neq i, i+1$
$h_i\widehat{h}_j(k)$	$h_iab^{[kp^j-1]}$	$2(p-1)p^i+2kp^j-2$	$0 \leq j,i; j \neq i, i+1$ $1 \leq k < p-1$
$\widehat{d}_i(k)$	$h_iab^{[kp^i+(p-1)p^{i-1}-1]}$	$2(p-1)(p^i+p^{i-1})$ $+2kp^i-2$	$i \geq 1, 1 \leq k \leq p-1$
$\widehat{k}_i(k)$	$h_iab^{[kp^{i+1}+p^i-1]}$	$2(k+1)p^{i+1}-2$	$i \geq 0, 1 \leq k < p-1$
$\widehat{p}_i(k)$	$h_iab^{[(p-1)p^{i+1}+(k+1)p^i-1]}$	$2(p-1)(p^i+p^{i+1})$ $+2(k+1)p^i-2$	$i \geq 0, 1 \leq k < p-1$

Theorem 3.2.5 (Chon-Nhu [18, Theorem 5.4])**.** *The Ext group $\mathrm{Ext}_A^{1,1+t}(\widetilde{H}^*(B\mathbb{Z}/p), \mathbb{F}_p)$ has an $\mathbb{F}_p$-basis consisting of all elements given by the following list*

1. $\alpha_0\widehat{h}_i = \left[\lambda_{-1}^0ab^{[(p-1)p^i-1]}\right], i \geq 1$;

2. $\alpha_0\widehat{h}_i(k) = \left[\lambda_{-1}^0ab^{[kp^i-1]}\right], i \geq 1, 1 \leq k < p-1$;

3. $\widehat{\alpha}(\ell) = \left[\lambda_{-1}^0ab^{[p+\ell]} + (\ell+1)\lambda_0^0ab^{[\ell+1]}\right], 0 \leq \ell < p-2$;

4. $h_i\widehat{h}_i(1) = \left[\lambda_{p^i-1}^1ab^{[p^i-1]}\right], i \geq 0$;

5. $h_i\widehat{h}_j = \left[\lambda^1_{p^i-1}ab^{[(p-1)p^j-1]}\right], i,j \geq 0, j \neq i, i+1;$

6. $h_i\widehat{h}_j(k) = \left[\lambda^1_{p^i-1}ab^{[kp^j-1]}\right], i,j \geq 0, j \neq i, i+1, 1 \leq k < p-1;$

7. $\widehat{d}_i(k) = (\mathcal{P}^0)^{i-1}\left(\left[\lambda^1_{p-1}ab^{[kp+p-2]}\right]\right), i \geq 1, 1 \leq k \leq p-1;$

8. $\widehat{k}_i(k) = (\mathcal{P}^0)^{i}\left(\left[\sum_{j=0}^{k}\frac{1}{j+1}\lambda^1_{j}ab^{[(k-j)p+j]}\right]\right), i \geq 0, 1 \leq k < p-1;$

9. $\widehat{p}_i(k) = (\mathcal{P}^0)^{i}\left(\left[\sum_{j=0}^{p-1-k}\frac{\binom{k+j}{j}}{j+1}\lambda^1_{j}ab^{[(p-j-1)p+k+j]}\right]\right), i \geq 0,\ 1 \leq k < p-1.$

Proposition 3.2.6 (Chon-Nhu [18, Proposition 5.5])**.** *The* $\mathrm{Ext}_A^{*,*}(\mathbb{F}_p,\mathbb{F}_p)$*-module* $\mathrm{Ext}_A^{s,*}(\widetilde{H}^*(B\mathbb{Z}/p),\mathbb{F}_p)$*, for* $s \leq 1$*, is generated by* $\widehat{h}_i$ $(i \geq 0)$*,* $\widehat{h}_i(k)$ $(i \geq 0, 1 \leq k < p-1)$*,* $\widehat{\alpha}(\ell)$ $(0 \leq \ell < p-2)$*,* $\widehat{d}_i(k)$ $(i \geq 1, 1 \leq k \leq p-1)$*,* $\widehat{k}_i(k)$ $(i \geq 0, 1 \leq k < p-1)$ *and* $\widehat{p}_i(k)$ $(i \geq 0, 1 \leq k < p-1)$ *subject only to the following relations*

- $h_i\widehat{h}_{i+1} = 0, i \geq 0;$
- $h_i\widehat{h}_{i+1}(k) = 0, i \geq 0, 1 \leq k < p-1;$
- $h_i\widehat{h}_i = 0,\ i \geq 0;$
- $h_i\widehat{h}_i(k) = 0,\ i \geq 0, 2 \leq k < p-1;$
- $\alpha_0\widehat{h}_0 = 0$*; and*
- $\alpha_0\widehat{h}_0(k) = 0, 1 \leq k < p-1.$

3.3 The image of the mod p Lannes-Zarati homomorphism on $\widetilde{H}^*(B\mathbb{Z}/p)$

The behavior of the Lannes-Zarati homomorphism $\varphi_s^{\widetilde{H}^*(B\mathbb{Z}/p)}$ for $s \leq 1$ is given by the following theorems.

Theorem 3.3.1 (Chon-Nhu [18, Theorem 5.6])**.** *The zero-th Lannes-Zarati homomorphism*

$$\varphi_0^{\widetilde{H}^*(B\mathbb{Z}/p)} : \mathrm{Ext}_A^{0,t}(\widetilde{H}^*(B\mathbb{Z}/p),\mathbb{F}_p) \longrightarrow (\mathbb{F}_p \otimes_A \mathscr{R}_0\widetilde{H}^*(B\mathbb{Z}/p))_t^{\#}$$

is an isomorphism.

Theorem 3.3.2 (Chon-Nhu [18, Theorem 5.7])**.** *The first Lannes-Zarati homomorphism*

$$\varphi_1^{\widetilde{H}^*(B\mathbb{Z}/p)} : \mathrm{Ext}_A^{1,1+t}(\widetilde{H}^*(B\mathbb{Z}/p),\mathbb{F}_p) \longrightarrow (\mathbb{F}_p \otimes_A \mathscr{R}_1\widetilde{H}^*(B\mathbb{Z}/p))_t^{\#}$$

sends

(i) $h_i\widehat{h}_i(1)$ *to* $\left[\beta Q^{p^i}ab^{[p^i-1]}\right]$*, for* $i \geq 0$*;*

(ii) $h_i\widehat{h}_j$ *to* $\left[\beta Q^{p^i}ab^{[(p-1)p^j-1]}\right]$ *for* $0 \leq j < i$*;*

(iii) $h_i\widehat{h}_j(k)$ *to* $\left[\beta Q^{p^i}ab^{[kp^j-1]}\right]$ *for* $0 \leq j < i$*,* $1 \leq k < p-1$*;*

(iv) $\widehat{k}_i(k)$ *to* $(\mathcal{P}^0)^i\left(\left[\beta Q^{k+1}ab^{[k]}\right]\right), i \geq 0, 1 \leq k < p-1$*; and*

(v) *others to zero.*

Corollary 3.3.3 (Chon-Nhu [18, Corollary 5.8])**.** *The first Lannes-Zarati homomorphism* $\varphi_1^{\widetilde{H}^*(B\mathbb{Z}/p)}$ *is not an epimorphism.*

3.4 The image of the mod 2 Lannes-Zarati homomorphism

The mod 2 Lannes-Zarati homomorphism has been carefully studied by many mathematicians for a long time. We summarize the results about the behavior of the mod 2 Lannes-Zarati homomorphism, which are pulished by Lannes-Zarati [72], Hung et al. [30], [25], [27], [32] in Proposition 3.4.1. We will also re-prove this proposition with a different approach.

Proposition 3.4.1 (Lannes-Zarati [72], Hung et al. [30], [25], [27], [32])**.**

(i) *The first mod 2 Lannes-Zarati homomorphism* $\varphi_1^{\mathbb{F}_2}$ *is an isomorphism.*

(ii) *The second mod 2 Lannes-Zarati homomorphism* $\varphi_2^{\mathbb{F}_2}$ *is an epimorphism.*

(iii) *The s-th mod 2 Lannes-Zarati homomorphism* $\varphi_s^{\mathbb{F}_2}$ *vanishes at all positive stems in* $\mathrm{Ext}_{\mathscr{A}}^{s,s+t}(\mathbb{F}_2,\mathbb{F}_2)$ *for* $3 \leq s \leq 5$*.*

Besides demonstrating the results were pulished about the behavior of the mod 2 Lannes-Zarati homomorphism on $\mathbb{F}_2$ $\varphi_s^{\mathbb{F}_2}$ for $1 \leq s \leq 5$. We also compute image of the indecomposable elements in $\mathrm{Ext}_{\mathscr{A}}^{6,6+t}(\mathbb{F}_2,\mathbb{F}_2)$ for $0 \leq t \leq 114$ through the sixth mod 2 Lannes-Zarati homomorphism.

Theorem 3.4.2 (Nhu [50, Theorem 1.1])**.** *The sixth mod 2 Lannes-Zarati homomorphism*

$$\varphi_6^{\mathbb{F}_2} : \mathrm{Ext}_{\mathscr{A}}^{6,6+t}(\mathbb{F}_2,\mathbb{F}_2) \longrightarrow \mathrm{Ann}((\mathscr{R}_6\mathbb{F}_2)^{\#})_t$$

is trivial on indecomposable elements in $\mathrm{Ext}_{\mathscr{A}}^{6,6+t}(\mathbb{F}_2,\mathbb{F}_2)$ *for* $0 \leq t \leq 114$*.*

In addition, using this method we can also check the results of Hung-Tuan in [34]

Proposition 3.4.3 (Hung-Tuan [34])**.**

(i) *The zero-th mod 2 Lannes-Zarati homomorphism* $\varphi_0^{\widetilde{H}^*(B\mathbb{Z}/p)}$ *is an isomorphism on* $\mathrm{Ext}_{\mathscr{A}}^{0,t}(\widetilde{H}^*(B\mathbb{Z}/p),\mathbb{F}_2)$*.*

(ii) *The first mod 2 Lannes-Zarati homomorphism* $\varphi_1^{\widetilde{H}^*(B\mathbb{Z}/p)}$ *is a monomorphism on* $\mathrm{Span}\{h_i\widehat{h}_j : i \geq j\}$ *and vanishes on* $\mathrm{Span}\{h_i\widehat{h}_j : i < j\}$*.*

(iii) *The s-th mod 2 Lannes-Zarati homomorphism* $\varphi_s^{\widetilde{H}^*(B\mathbb{Z}/p)}$ *vanishes in all positive stems in* $\mathrm{Ext}_{\mathscr{A}}^{s,s+t}(\widetilde{H}^*(B\mathbb{Z}/p),\mathbb{F}_2)$ *for* $2 \leq s \leq 4$*.*

Remark 3.4.4. By Proposition 2.5.1, it is easy to verify that $(\mathbb{F}_2 \otimes_{\mathscr{A}} \mathscr{R}_1 \widetilde{H}^*(B\mathbb{Z}/p))^{\#}$ is spanned by

$$\left\{\left[Q^{2^i-1}b^{[2^j-1]}\right] : i \geq j\right\} \cup \left\{(Sq^0)^i \left(\left[Q^{2(2^j-1)}b^{[1]}\right] + \left[Q^{2^{j+1}-1}b^{[2]}\right]\right) : i \geq 0, j \geq 1\right\}.$$

Therefore, the first Lannes-Zarati homomorphism $\varphi_1^{\widetilde{H}^*(B\mathbb{Z}/p)}$ is not an epimorphism.

Conclusion

In this thesis. we achieved the following main results:

1. We construct the chain-level representation of the dual of the mod p Lannes-Zarati homomorphism $(\varphi_s^M)^\#$ on Singer-Hung-Sum chain complex as well as the chain-level representation of the mod p Lannes-Zarati homomorphism φ_s^M on the complex $\Lambda \otimes M^\#$, for any $\mathscr{A}$-module M (see Proposition 2.2.5). These results will be used to find kernel and image of the mod p Lannes-Zarati homomorphism φ_s^M with s small for p odd.

2. We develop the power operation $\mathcal{P}^0$ acting on $\mathrm{Ext}_{\mathscr{A}}^{s,s+t}(\mathbb{F}_p, \mathbb{F}_p)$ (see Liulevicius [41], [42] và May [19]). For $M = \mathbb{F}_p$ and $M = \widetilde{H}^*(B\mathbb{Z}/p)$, we showed that there is the operation $\mathcal{P}^0$ that acts on $\mathrm{Ext}_{\mathscr{A}}^{s,s+t}(M, \mathbb{F}_p)$ and on $(\mathbb{F}_p \otimes_{\mathscr{A}} \mathscr{R}_s M)^\#$. Moreover, this operation is also commutative with the Lannes-Zarati homomorphism φ_s^M (see Proposition 2.4.3). This makes reduce significantly the computation. Therefore, this operator becomes an important tool for studying the behavior of the mod p Lannes-Zarati homomorphism.

3. Investigate the behavior of the mod p Lannes-Zarati homomorphism φ_s^M for $M = \mathbb{F}_p$ and $M = \widetilde{H}^*(B\mathbb{Z}/p)$. As a result, we obtained complete image of $\varphi_s^{\mathbb{F}_p}$ with $1 \leq s \leq 3$ (see Theorem 3.1.1, Theorem 3.1.2, Theorem 3.1.4) and image of $\varphi_s^{\widetilde{H}^*(B\mathbb{Z}/p)}$ with $s = 0, 1$ (see Theorem 3.3.1, Theorem 3.3.2).

4. Finally, we verify that the results of the mod 2 Lannes-Zarati homomorphism have been published in the literature [72], [25], [27], [32], [30]. The obtained results are similar to the published results but with simpler calculation (see Proposition 3.4.1, Proposition 3.4.3). Based on the results of Chen [12] on the indecomposable elements of $\mathrm{Ext}_{\mathscr{A}}^{6,6+t}(\mathbb{F}_2, \mathbb{F}_2)$ for $0 \leq t \leq 114$, we compute the image of these elements through the sixth Lannes-Zarati homomorphism for $p = 2$ using a different approach, that is, we do not use the result of the " hit " problem on $\mathscr{D}_6$ (see Theorem **??**).

We will be of interest to study some following problems:

1. We will continue to study the behavior of the mod p Lannes-Zarati homomorphism for $s \geq 4$.

2. We plan to study the mod p Singer tranfer where p is an odd prime.

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List of author's papers related to the thesis

(1) Phan Hoang Chon, Pham Bich Nhu (2019), "On the mod p Lannes-Zarati homomorphism", *Journal of Algebra.*, **537**, 316-342.

(2) Phan Hoang Chon, Pham Bich Nhu (2020), "The cohomology of the Steenrod algebra and the mod p Lannes-Zarati homomorphism", *Journal of Algebra.*, **556**, 656-695.

(3) Pham Bich Nhu (2020), "On behavior of the sixth Lannes-Zarati homomorphism", *East-West Journal of Mathematics.*, **22**, no.1, 1-12, https://doi.org/10.36853/ewjm.2020.22.01/01